A SELF-CONSISTENT TIDAL THEORY FOR IMPERFECTLY ELASTIC BODIES.

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Planetary scientists have long recognized that the most interesting consequences of tides arise only if imperfect elasticity (energy dissipation) is present in either the planet or its orbiting satellite. For example, without the delayed response associated with anelastic behavior, orbital evolution [1] does not occur since then the energy and angular momentum of the orbit are unchanged. Similarly, tidal heating [2,3], which under certain circumstances could be a significant heat source for the past Moon [4], requires a phase lag between stress and strain if there is to be a net deposition of energy over a cycle.

Even though energy loss is crucial for these processes to operate, only Darwin [5] has constructed a self-consistent model for the deformation, stress and mechanical heating histories of a tidally distorted, imperfectly elastic body. Other approaches to the tidal problem artifically introduce dissipation by analogy [6,7] with a linear, mass-spring-dashpot system driven by an oscillating force. Such a system responds with a phase lag, proportional to the energy-dissipating term and the forcing frequency, as long as the damping is small and the oscillations are much slower than the resonant frequency; a similar response is found for a purely viscous or an elastico-viscous Earth [5]. Most previous attacks on the tidal problem [e.g., 6-8] have thus assumed that the tidal deformations are delayed either in time or space, suffering a phase lag much like the damped oscillator. While this is a reasonable first approximation when materials are not especially dissipatory, in general such a model is not self-consistent, i.e., the expressions for the stress and strain tensors, independently arrived at, do not satisfy the constitutive equations, as they must if correct. In the usual method then the phase lag of each term in the tidal strain is set equal to Q_f^{-1} (the specific dissipation function measured at frequency f). Calculations of the orbital evolution [1] as such view the tidal bulge as delayed through the angle Q^{-1} and, in this way, compute the perturbing forces on the orbit; in practice, however, the observed evolution defines Q for the tidal problem because the tidal Q cannot be otherwise easily measured. Moreover, with this model the heating accompaning tidal flexing is taken as merely Q^{-1} times the stored strain energy. The difficulty with both calculations is that a patched-up elastic solution is used whereas both phenomena are inherently anelastic. There has also been one other attempt [5,9] at calculating tidal heating; it is self-consistent but treats the Moon as a viscous fluid, surely a questionable model.

We here summarize a self-consistent theory for the tidal deformation and dissipation of an imperfectly elastic body; the complete derivation of the fairly complex analytic expressions is presented elsewhere [10].

The two simplest models [6] illustrating the anelasticity of real materials are a Maxwell material (represented by a spring and dashpot in series; earlier considered by Darwin [5]) and a Kelvin-Voight material (spring and dashpot in parallel); a perhaps more realistic, but considerably more complex, model is that for a fluid-saturated cracked solid [11]. None of these is to be preferred in general; nevertheless for reasons of mathematical tractability we work with the Kelvin-Voight model. This will permit us to formally solve the anelastic tidal problem and to compare such a solution against previous elastic results.

The constitutive relation for the stress $\underset{\approx}{T}$ in an isotropic, homogeneous, incompressible, Kelvin-Voight material is

$$\underset{\approx}{T} = -p\underset{\approx}{I} + 2\mu\underset{\approx}{\varepsilon} + 2M\dot{\underset{\approx}{\varepsilon}} \quad , \tag{1}$$

where p is the arbitrary pressure required by the incompressibility assumption, $\underset{\approx}{I}$ is the identity, μ is the shear modulus while its viscous counterpart is M; the strain $\underset{\approx}{\varepsilon}$ is the symmetric part of the displacement gradient $\nabla \underset{\sim}{u}$ and $\dot{} \equiv \frac{\partial}{\partial t}$. Under the assumption that the displacement gradients are small compared to $\underset{\approx}{I}$, the displacements are determined by the equation

$$\nabla^2 \dot{\underset{\sim}{u}} + \alpha \nabla^2 \underset{\sim}{u} + \nabla(\rho V - p) = 0 \quad , \tag{2}$$

where ρ is the density, V is the total gravitational potential and $\alpha^{-1} \equiv M/\mu$ is the material's characteristic response time. V is the sum of the external body's perturbing potential W and the deformed body's self-gravitational potential (which is that of an initially undeformed sphere plus S, that due to a surface layer of thickness $u_r(r = R_o)$, the radial displacement at the outer boundary). Equation (2) is subject to the boundary condition that the tractions vanish on the deformed surface. Integrating (2) with respect to time,

$$\nabla^2 \underset{\sim}{u} = \nabla Q \quad \text{with} \quad Q = -\mu^{-1} e^{-\alpha t} \int_o^t (\rho V - p) e^{\alpha \tau} d\tau \quad . \tag{3}$$

Thus

$$\underset{\sim}{u} = \sum_{n=o}^{\infty} (A_n R^2 \nabla Q_n + B_n Q_n \underset{\sim}{R} + \nabla \xi_n) \tag{4}$$

with $A_n = (3+n)/[2(2n+3)(n+1)]$, $B_n = -n/[(2n+3)(n+1)]$; Q_n and ξ_n are the elements in spherical harmonic expansions of Q and of an arbitrary harmonic function ξ coming from the boundary conditions. The specification of V, along with the evaluation of ξ and p from the boundary conditions, complete the solution.

Using the same technique as for the elastic problem, we determine that the displacements in spherical coordinates are

$$u_R = (\rho/RM) e^{-\alpha t} \sum_{n=2}^{\infty} \int_o^t \{W_n [n\hat{\gamma}_n R_o^2 - (nA_n + B_n)(1+\gamma_n) R^2] - S_n [n\hat{\Gamma}_n R_o^2 - (nA_n + B_n)(\Gamma_n - 1) R^2\} e^{\alpha \tau} d\tau \quad , \tag{5}$$

$$u_\theta = (\rho/RM) e^{-\alpha t} \sum_{n=2}^{\infty} \int_o^t \{\frac{\partial W_n}{\partial \theta} [\hat{\gamma}_n R_o^2 - A_n (1+\gamma_n) R^2] - \frac{\partial S_n}{\partial \theta} [\hat{\Gamma}_n R_o^2 - A_n (\Gamma_n - 1) R^2]\} e^{\alpha \tau} d\tau \quad , \tag{6}$$

$$u_\Phi = (\rho/RM) e^{-\alpha t} \sum_{n=2}^{\infty} \int_o^t (\sin\theta)^{-1} \{\frac{\partial W_n}{\partial \Phi} [\hat{\gamma}_n R_o^2 - A_n (1+\gamma_n) R^2] - \frac{\partial S_n}{\partial \Phi} [\hat{\Gamma}_n R_o^2 - A_n (\Gamma_n - 1) R^2]\} e^{\alpha \tau} d\tau \quad . \tag{7}$$

The parameters γ_n, $\hat{\gamma}_n$, Γ_n, $\hat{\Gamma}_n$ and ψ_n (needed below) are constants expressed in terms of n, A_n and B_n. The velocity field is of the same form as (5-7) except that W_n and S_n are replaced by $\dot{W}_n$ and $\dot{S}_n$. The strain and strain rate come from substituting the displacements, given in (5-7), into

$$\underset{\approx}{\varepsilon} \equiv \frac{1}{2}(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T) \tag{8}$$

and its time derivative; ^{T} denotes the transpose.

The connection between our solution and that for the corresponding linearly elastic solution is most clearly seen in the surface potential, whose n^{th} component can be found to be

$$S_n = k_n W_n - k_n e^{-\beta_n t} \int_0^t W_n e^{\beta_n \tau} d\tau \quad , \tag{9}$$

where k_n is the nth Love number and $\beta_n = (1+\psi_n/\tilde{\mu})\alpha$ with $\tilde{\mu}$ the nondimensional rigidity. The second term represents the contribution due to the viscosity; this vanishes as $M \to 0$ and thus the classical result, as expressed by the first term, stands alone. Substituting (9) into (5), (6) and (7) shows that the displacements decompose similarly into an elastic part and a viscous part. Encouragingly, these expressions, at least for $n = 2$, reduce to ones found in the elastic solution [8,12] when $M \to 0$.

The rate of mechanical heating of the body during its tidal deformation can be shown to be the internal dissipation rate

$$D = 2M\dot{\underset{\approx}{\varepsilon}} \cdot \dot{\underset{\approx}{\varepsilon}} \quad , \tag{10}$$

which then appears as a source term in the initial-boundary value problem for the absolute temperature. The validity of the above analysis is restricted to a range of temperatures for which material parameters and pressure are essentially constant, since all temperature dependence has been neglected. Moreover, the solution suffers in that experimental values of M have not been measured for geologic materials. Regardless of these failings, the above viscoelastic solution does provide a self-consistent, analytical characterization of dissipation for a tidally-interacting object composed of imperfectly elastic material.

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