A Streamline Model of Horseshoe Torque Saturation. **Wm. R. Ward, Southwest Research Institute, Boulder, CO 80302**

At a Lindblad resonance in an optically thin, non-interacting disk of particles, the particles librate, and so too do their angular momenta. As a result, the cumulative torque from the disk eventually decays to zero due to phase mixing of the particles [1]. Here we show that a similar behavior is manifested in the horseshoe orbit zone. Exchange of angular momentum between particles and perturber causes the well-known horseshoe behavior for nearby particles and a drag on the orbit of the perturber [2-4]. Using a simple streamline model, with $i(o)$ designating the inner (outer) leg of the horseshoe trajectory, the torque for an individual streamline can be written [4],

$$\delta T = r_o B_o |\Omega_s - \Omega_o| \delta r_o (\sigma_o/B_o - \sigma_i/B_i)(r_o^2\Omega_o - r_i^2\Omega_i) \quad (1)$$

The total torque is found by summing over all streamlines. Integrating out to the horseshoe zone edge, the initial cumulative torque is [4]

$$T \approx 4\sigma r^4 |A| B w^4 \left(\frac{d\ \ln(\sigma/B)}{d\ \ln r} \right) \quad (2)$$

where w is the zone half-width normalized to r, $\Omega(r)$ and Ω_s are the mean motions of a particle and the secondary, $\sigma(r)$ is the surface density of the disk and $A = (r/2)d\Omega/rdr$, $B \equiv (2r)^{-1}d(r^2\Omega)/dr$ are Oort constants.

The torque described by eqn (2) is only temporary. Since horseshoe orbits leading and trailing the secondary are connected, a particle initially interior to the perturber and approaching from the rear will be turned back along an exterior path. In a time $P/2$, where $P(x) \sim 2\pi/x|A|$ is the horseshoe libration period and x is the normalized distance to the perturber, this particle will re-encounter the secondary approaching from the front and suffer the reverse fate. Thus, δT will undergo sign reversals at intervals $\Delta t = P/2$. To model this, a step function $\epsilon(2t/P) = \pm 1$ can be introduced into eqn (1) that turns positive (negative) whenever $2t/P = n$, where n is an even (odd) integer. Since the outermost streamlines have the shortest libration periods, the position of the n^{th} reversal, $x_n = w[nP(w)/2t]$, propagates inward with time. In terms of a dimensionless time variable, $\tau = 2t/P(w)$, the width of an annulus, $x_{n+1} - x_n \equiv \Delta x = w/\tau$, narrows as the number of annuli increases. Since positive and negative annuli oppose each other, the torque decays as this mutual interference becomes more complete.

We define a saturation function,

$$S(\tau) \equiv \frac{4}{w^4} \int_{-0}^{w} \epsilon(x,t) x^3\, dx \quad (3)$$

to be used as a multiplier of eqn (2): $T(\tau) \to S(\tau)T$. Fourier decomposing the step function,

$$\epsilon(t,x) = \frac{4}{\pi} \Sigma_{odd} \frac{1}{n} \sin(n|A|xt)$$

and then integrating eqn (3) gives

$$S(\tau) = \frac{16}{\pi} \Sigma_{odd} \frac{1}{n} \left[\left(\frac{3}{(n\pi\tau)^2} - \frac{6}{(n\pi\tau)^4} \right) \sin(n\pi\tau) \right.$$

$$\left. - \left(\frac{1}{(n\pi\tau)} - \frac{6}{(n\pi\tau)^3} \right) \cos(n\pi\tau) \right].$$

This expression rapidly converges (in order) and is shown in Fig. 1a. As $\tau \gg 1$, $S(\tau) \to -4\tau^{-1}[(4/\pi^2)\Sigma_{odd} n^{-2} \cos(n\pi\tau)]$; the bracketed quantity is simply the Fourier representation of a sawtooth function of amplitude ½.

The angular momentum exchanged with

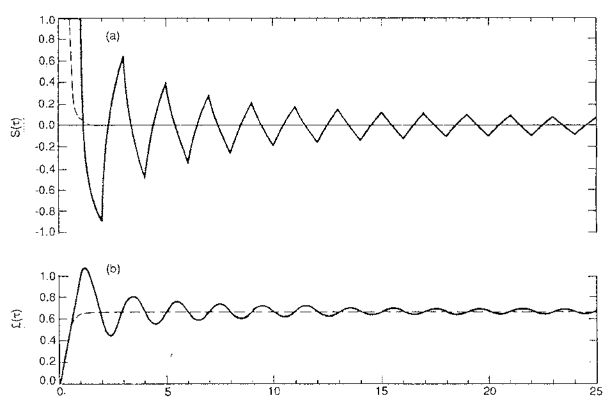

Figure 1. (a) The torque saturation function due to phase mixing of streamlines. (b) The normalized angular momentum exchanged between the perturber and disk. The dashed curves are effective secular portions using $S_{sec} = 1$ when $x < ½$ and $S_{sec} = (2\tau)^{-4}$ when $x > ½$.

the perturber is obtained by a time integration of the torque: $\Delta L = \mathcal{L}(\tau)\cdot T\cdot P(w)/2$, where

$$\mathcal{L}(\tau) \equiv \int_0^{\tau} S(\tau)d\tau = = \frac{16}{\pi^2}\Sigma_{odd}\frac{1}{n^2}\left[\frac{1}{3} - \left(\frac{1}{n\pi\tau} - \frac{2}{(n\pi\tau)^3}\right)\sin(n\pi\tau) - \frac{2}{(n\pi\tau)^2}\cos(n\pi\tau)\right]$$

and is plotted in Fig 1b. This quantity quickly approaches a limiting value, $\mathcal{L}(\infty) = (16/3)\pi^{-2}\Sigma_{odd}(1/n^2) = 2/3$. The total angular momentum exchanged with the perturber as $t \to \infty$ is thus

$$\Delta L_T = \frac{8}{3}\pi\sigma r^4 B w^3\left(\frac{d\ln(\sigma/B)}{d\ln r}\right).$$

At a distance $r_o = r_s(1+x)$ outside the perturber's orbit, the density

$$\sigma(r_o,t) = \frac{1}{2}(\sigma_o + \sigma_i') + \frac{1}{2}\epsilon(x,t)(\sigma_o - \sigma_i')$$

jumps between $\sigma_o(r_o)$ and $\sigma_i'(r_o)$ at intervals $P(x)/2$ as the streamline's original inner and outer legs are repeatedly exchanged. Here, $\sigma_i'(r_o)$ is the surface density of initially interior particles after encounter has perturbed them to the outside track. This is *not* in general the original density, $\sigma_i(r_i)$. To see this, we apply the constraint that the mass flux remains unchanged as one follows a particle along the streamline so that

$$F_i = \sigma_i(r_i)r_i|\Omega_i - \Omega_s|\delta r_i = \sigma_i'(r_o)r_o|\Omega_o - \Omega_s|\delta r_o$$

Particles from opposite edges of a given streamline have Jacobi constants that differ by $|\delta J| = \delta r|dJ/dr| = 2rB|\Omega_s - \Omega|\delta r$ [4]. This difference must persist after encounter, which allows us to deduce the change in streamline width going from one leg of the horseshoe trajectory to the other. This implies that $\sigma_i'(r_o) = \sigma_i(r_i)\cdot(B_o/B_i)$.

Eqn (4) can be averaged over some radial scale, $d \ll w$, neglecting slow changes in σ_o and σ_i';

$$\langle\sigma_o\rangle = \frac{1}{d}\int_{x-d/2}^{x+d/2}\sigma(x)\,dx = \frac{1}{2}(\sigma_o + \sigma_i') + \frac{1}{2}D(x,\tau)(\sigma_o - \sigma_i')$$

where

$$D(x,\tau) = \frac{8}{\pi^2}\left(\frac{w}{d}\right)\frac{1}{\tau}\Sigma_{odd}\frac{1}{n^2}\sin(n\pi\tau x/w)\sin(n\pi\tau d/2w).$$

If τ is small enough that $\theta_n \equiv n\pi\tau d/2w \ll 1$ for n up to some $N >> 1$, the first $n < N$ terms of the quantity, $D(x,\tau)$, are approximately

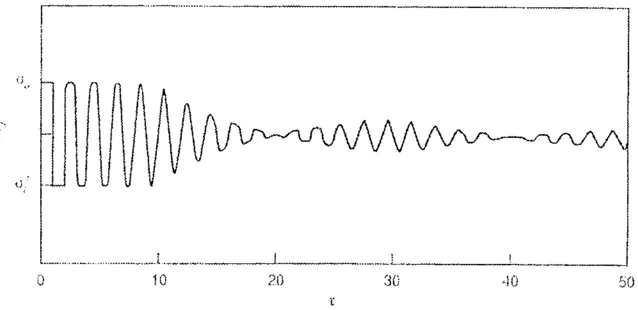

Figure 2. Averaged surface density over a radial interval d as a function of time.

$$\frac{4}{\pi}\Sigma_{odd}^{N}\frac{1}{n}\sin(n\pi\tau x/w) \approx \epsilon(x,\tau)$$

i..e., the step function is nearly regenerated. However, as τ increases, the value of N decreases until for $\tau \geq w/d$, $\theta_n \geq O(1)$ for all n. Past this point, $D(x,\tau)$ exhibits a beat pattern with nodes at intervals of $\Delta\tau = 2(w/d)$ and a magnitude that decays as $1/\tau$ as shown in figure 2. Hence, the ring becomes mixed on a length scale d in a time $\tau_d \sim O(w/d)$ and for $\tau \gg \tau_d$, the density averaged over the scale d approaches $\langle\sigma_o\rangle \to (1/2)(\sigma_o + \sigma_i') = (1/2)(\sigma_o + \sigma_i B_o/B_i)$. The complimentary average for the interior density, $\langle\sigma_i\rangle$, is obtained by reversing subscripts. Consequently,

$$\frac{\langle\sigma_o\rangle}{B_o} = \frac{\langle\sigma_i\rangle}{B_i} = \frac{1}{2}\left(\frac{\sigma_o}{B_o} + \frac{\sigma_i}{B_i}\right)$$

and after mixing, the quantity, $\langle\sigma\rangle/B$, assumes the same value on both interior and exterior legs of each horseshoe orbit and the torque vanishes.

Complete saturation of the torque can be prevented by a finite viscosity, ν [5]. If the viscous timescale $\tau_\nu \sim w^2/\nu < P(w)$, diffusion from a surrounding disk will try to reestablish the global gradient, and the torque decays to a non-zero value that depends on the ratio of these time scales [6].

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