Photometric observations are the primary source of information on the bulk of the asteroid population. The macroscopic shapes, rotational states, and scattering parameters of asteroids can be deduced from photometric measurements of total brightness in different viewing/illumination geometries. Shape classes include convex, nonconvex, and binary objects, while rotational states include relaxed rotation, precessing motion, and mutually orbiting configurations. We discuss the role of lightcurve inversion in light of results from other sources.

1. INTRODUCTION

Disk-resolved images can be obtained of only a limited number of the small atmosphereless bodies of our solar system. This is why disk-integrated photometry, i.e., measurements of total brightness, is, and will remain, a major source of information on these objects. The total brightness of an object as a function of time (as the viewing/illumination geometry changes) is called its lightcurve; the parameters to be determined from lightcurve observations are the object’s shape, its rotational state, and the scattering properties of its surface.

The problem of lightcurve inversion has been studied for nearly a century; however, restricted choices of observing geometry and scattering law in the early analytical studies have caused erroneously negative preconceptions about the uniqueness and stability of the solution (Russell, 1906). Partly because of this, and partly because of the relative scarcity of accurate lightcurve observations, the shape models used in lightcurve inversion until recently have been extremely simple — typically triaxial ellipsoids or their modifications — while the rotational states have usually been heuristically determined from certain characteristics of the lightcurves. Such traditional methods are well described in, e.g., Magnusson et al. (1989, 1996), Barucci et al. (1992), Kwiatkowski (1995), and Binzel et al. (1993); they are practical if the data are very limited. We encourage the reader to study the Asteroids II chapter by Magnusson et al. (1989): The methods described there are especially suitable for getting the first pole-direction estimate for an asteroid not yet well observed.

The view taken in this chapter is that lightcurve inversion is a typical representative of a problem category known as inverse problems. These are solved with modern deconvolution methods and optimization techniques. By an asteroid model we mean a general physical model not based on a shape given a priori. The model must also utilize all available (photometric) data in the analysis, not just, e.g., amplitudes or epochs of lightcurve features. The first general analysis of the inverse problem, including uniqueness theorems for convex shapes, is offered in Kaasalainen et al. (1992a,b). A robust and generally applicable inversion scheme is developed in Kaasalainen and Torppa (2001) and M. Kaasalainen et al. (2001). Models for precessing and binary systems have recently been developed as well (Kaasalainen, 2001; Mottola and Lahulla, 2000; Pravec and Hahn, 1997).

A number of spacecraft and radar images of asteroids have been obtained during the last decade (see Chapman, 2002; Cheng, 2002; Ostro et al., 2002; and references therein). The most conspicuous feature in these is the existence of substantial irregularities: sharp edges, completely asymmetric shapes, globally significant valleys and indentations, and large craters. The number and accuracy of groundbased lightcurve measurements have also increased significantly, so it is important to know how much information can really be obtained from photometry in principle and how well the results can be expected to describe the real objects.

In section 2, we discuss the fundamental role of lightcurve inversion in light of (and compared with) the results obtained with other groundbased and in situ methods during the last decade. In section 3, we address some generic concepts of lightcurve analysis, while section 4 is devoted to the discussion of convex inversion and the corresponding stability properties. We discuss the methods and limitations of non-convex inversion in section 5, while section 6 is concerned with the role of the light-scattering model of the surface material. The determination of the rotational state is the subject of section 7, and binary models are discussed in section 8. Finally, in section 9, we list some conclusions and topics for future work. Readers who want to form a quick overview of the subject can skip sections 4–6 on first reading.
2. GROUND TRUTH AND THE ROLE OF LIGHTCURVE INVERSION

The shapes of asteroids result from (1) the accretion process in the swarm of planetesimals or (2) the (repeated) fragmentation of a parent body caused by a catastrophic collision, followed in some cases by (3) a possible gravitational reaccumulation into a “rubble pile” structure. Single fragments are mostly irregular in shape, while the remnants of the original planetesimals and large rubble piles can mostly be expected to have more regular shapes, close to rotational equilibrium shapes, i.e., biaxial and triaxial ellipsoids (Farinella et al., 1981; Holsapple et al., 2002; Richardson et al., 2002; Scheeres et al., 2002). Continuous exposure to the space environment (crater and regolith formation, microimpact erosion, and bombardment by charged particles) affects the surfaces of asteroids by changing their global morphology as well as their macroscopic and microscopic photometric properties.

Modeling the shapes and global surface properties of asteroids from disk-integrated photometric data necessitates robust and flexible inversion methods. The foremost objective in establishing the degree of reliability is the mathematical assurance of the uniqueness and stability of the solution, as discussed in the sections below. The reassuring knowledge that we are on firm ground instead of thin ice is important, but it is also necessary to check the adopted inversion method against reality. This can be done if the shape and surface properties of the body that originated a given set of lightcurves are known. This “calibration” has been explored both theoretically and through laboratory simulations (e.g., Barucci and Fulchignoni, 1985; M. Kaasalainen et al., 2001).

The images obtained by spacecraft, orbiting observatories, radar observations, or adaptive optics show that the larger asteroids (Ceres, Vesta) have relatively regular global shapes, close to a spheroid or equilibrium ellipsoid, while the medium- and small-sized objects have more irregular shapes, characterized by sharp edges, concavities, and peculiar features that dominate the global morphology (e.g., the saddle-shaped feature Himeros on Eros). The shapes and topography of such closely observed objects can be represented with accurate numerical models (Simonelli et al., 1993) such as the one shown in Fig. 1. The obtained models allow us to calculate the volume of the body and estimate its density, provided that a mass value is available from mutual perturbations of asteroidal orbits, spacecraft trajectories, or the presence of a satellite (Hilton, 2002). The knowledge of the shapes of these bodies is essential for modeling their internal structure, if other data (spectrometry, altimetry, gravimetry, mineralogical composition, etc.) are available. Moreover, such cartography can be used to interpret an asteroid’s surface features in terms of its evolution.

The availability of the disk-resolved dataset on the asteroids 951 Gaspra (Thomas et al., 1994), 243 Ida (Thomas et al., 1996), and 433 Eros (Veverka et al., 2000), taken during the Galileo and NEAR Shoemaker missions, represents the ground truth that allows us to understand the limits of the adopted inversion methods. All these objects have been well observed photometrically; the corresponding detailed inversion results are discussed in M. Kaasalainen et al. (2001). The global triaxial dimensions of the lightcurve inversion and probe models agree within 5–10% for each of the three bodies, and the global-scale details of the two shape models resemble each other closely. The spin-vector directions and rotation rates also completely agree (within a few degrees) with the spacecraft data for Eros and Gaspra; the restricted observation geometries for Ida allow two possible pole solutions, one of which closely agrees with the correct pole. Lightcurve inversion results also closely agree (M. Kaasalainen et al., 2001) with the radar-based models for 1620 Geographos (Hudson and Ostro, 1999) and 6489 Golevka (Hudson et al. 2000).

Traditional pole determination methods and their variants have also produced good estimates of the pole directions and rotation periods for most of the “test-case” asteroids above (see the corresponding references above and references therein). These estimates are particularly useful as initial values that can be refined with the techniques discussed in this chapter. Traditional pole determination schemes are often unable to resolve pole ambiguities (section 7), and irregularly shaped asteroids may confuse such methods [cf. the analysis of 6489 Golevka by Mottola et al. (1997b)].

Figure 1 shows the topographic model of 951 Gaspra, based on Galileo data, while Fig. 2 shows the corresponding model from lightcurve inversion at the same viewing geometry (30°N of the equator and 60° off the long axis of the body). The two figures well demonstrate three important points (each discussed in more detail in the sections below): (1) Lightcurve inversion provides a good model of the global shape of the object. (2) The stable (most reliable) shape

![Fig. 1. A view of the topographic model of 951 Gaspra, based on Galileo data.](image-url)
3. DIRECT VS. INVERSE PROBLEM

The total brightness $L$ of an object can be written as $L = L(\mathcal{R}, \mathcal{S}, \mathcal{D}; \mathcal{O}, \mathcal{A}; t)$, where $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{D}$ are the sets of parameters describing the shape, scattering properties, and rotational state of the object respectively. The orbital parameters describing the motion of the observer and the asteroid are given by $\mathcal{O}$ and $\mathcal{A}$, so the evolution of $L$ in time $t$, i.e., the asteroid’s lightcurve, is completely determined by the five parameter sets.

Lightcurves produced by arbitrary objects can be computed numerically using a ray-tracing code. Second-order scattering can usually be neglected, so it is straightforward to write a multipurpose procedure that checks which parts of the surface are visible to both Earth and the Sun. To this end, the surface is best given as a polyhedron with small triangles (on the order of 1000) as facets; precomputed lists of possible ray-blockers for each facet make the computation fast. This tessellation can be applied separately to each object of a group; thus binary objects can be handled with the same code (suitably modified to include time dependence if the rotation is nonsynchronous).

Once it is known that a surface patch $ds$ is both visible and illuminated, its contribution $dL$ to the total brightness is given by (omitting irrelevant scale factors such as the squares of distances)

$$dL = S(\mu, \mu_0) \sigma \ ds$$ (1)

where $\sigma$ and $S$ are albedo and the scattering law (in a simple form here; more arguments can naturally be included, as in the full Hapke model), $\mu = \mathbf{E} \cdot \mathbf{n}$ and $\mu_0 = \mathbf{E}_0 \cdot \mathbf{n}$, where $\mathbf{E}$ and $\mathbf{E}_0$ are, respectively, unit vectors toward the observer (Earth) and the Sun, and $\mathbf{n}$ is the surface unit normal. Lambert’s law, for example, is $S_L = \mu \mu_0$, while the Lommel-Seeiger law is $S_{LS} = S_L/(\mu + \mu_0)$. Note that $L$ is given in intensity units rather than magnitudes. The separate expression $\sigma$ for albedo as a scale factor representing the local darkness of the surface is intuitively obvious for sufficiently large surface patches. The complete physical description of the small-scale structure represented in the scattering law $S$, with the albedo of the surface material as one parameter, is much more complicated. It should be stressed that “albedo” is, indeed, a somewhat vaguely defined concept, and our $\sigma$ is not the Bond, the geometric, or the single-scattering albedo [often denoted by $w$ or $\sigma_0$; see Bowell et al. (1989)].

The direct problem of lightcurve computation is straightforward: Just add together all the relevant $dL$ contributions from equation (1) to obtain the total brightness $L$ at any given geometries. However, the inverse problem is obviously difficult: The effects of $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{D}$ are combined in $L$, and now the sets should be disentangled. A general solution can only be found when all parameters are determined simultaneously. Our aim is thus to minimize

$$\chi^2 = \|L_{obs} - L(\mathcal{R}, \mathcal{S}, \mathcal{D})\|^2$$ (2)
where \( \mathbf{L}_{\text{obs}} \) and \( \mathbf{L} \) are vectors containing the observed and modeled brightnesses respectively at the observation epochs.

An important aspect of the scattering law is its strong dependence on the solar phase angle \( \alpha = \text{acos} (\mathbf{E} \cdot \mathbf{E}_0) \). If this cannot be modeled accurately, fitting absolute brightnesses is useless. On the other hand, absolute brightnesses of observations are often not known accurately enough in any case. To obtain proper line-over-points fits, the only possibility is usually to regard some or all of the photometric data as relative. In this case we minimize

\[
\chi^2_{\text{rel}} = \sum \frac{\left( \frac{\mathbf{L}_{\text{obs}}(i)}{\chi^2_{\text{obs}}} - \frac{\mathbf{L}(i)}{\chi^2_{\text{obs}}} \right)^2}{\chi^2_{\text{obs}}(i)}
\]

where, through the average brightnesses \( \bar{\mathbf{L}}(i) \) of each lightcurve sequence \( i \), both the observed and the model lightcurves are renormalized to mean brightnesses of unity. This corresponds to leaving an offset magnitude for each lightcurve as a free parameter to be determined; equation (3) is more advantageous as it discards all scale factors and thus keeps the number of free parameters as low as possible. Relative photometry is slightly more insensitive to the flattening of the body shape in the direction of the rotation axis than absolute photometry (for example, a bare spheroid can obviously be mistaken for a sphere if only relative photometry is available). This naturally emphasizes the role of good polar-aspect coverage.

Using equation (3) reflects a natural phenomenon: It is primarily the shapes of the lightcurves that are strongly connected with the pole, the period, and the shape of the asteroid. The absolute brightnesses are principally connected with the scattering properties, thus one can actually decouple one set of parameters from the rest to some extent.

4. CONVEX MODELS

4.1. Representations of a Convex Surface

The inverse problem becomes analytically tractable if it is assumed that the body can be modeled with a convex shape. Such a shape is computationally easiest to give in the form of a convex polyhedron. Define the matrix \( \mathbf{M} \) by

\[
\mathbf{M}_{ij} = S_j(\mu^{(ij)}, \mu_0^{(ij)}) \mathbf{n}_j
\]

where \( S_j \) and \( \mathbf{n}_j \) are the scattering law and albedo at the facet \( j \); \( \mu^{(ij)} = \mathbf{E}_i \cdot \mathbf{n}_j \) and \( \mu_0^{(ij)} = \mathbf{E}_{\text{obs}} \cdot \mathbf{n}_j \) for the observation \( i \) (in the asteroid’s frame of reference); and \( \mathbf{n}_j \) is the chosen surface outward normal of the facet \( j \). If either \( \mu^{(ij)} \) or \( \mu_0^{(ij)} \) is less than or equal to zero, \( \mathbf{M}_{ij} \) vanishes. From equation (1) we immediately have

\[
\mathbf{L} = \mathbf{M} \mathbf{s}
\]

where the vector \( \mathbf{s} \) contains the areas of the facets of the polyhedron. These must obviously be positive; the easiest way to guarantee this is to represent each \( s_j \) exponentially, the optimization parameter being now the exponent \( a_j \)

\[
s_j = \exp(a_j)
\]

Since the number of fitted parameters must be large (on the order of 1000) to make sure that the result does not depend on the directions of the surface normals, the conjugate-gradient method (see, e.g., Press et al., 1992) is efficient for minimizing \( \chi^2_{\text{rel}} \). Once the areas of the facets are known, the vertices of the facets can be obtained straightforwardly by iteratively solving the so-called Minkowski problem (Kaasalainen and Torppa, 2001; Lamberg, 1993; Lamberg and Kaasalainen, 2001).

It is also possible to use the more general concept of curvature function given as a smooth function series (Kaasalainen and Torppa, 2001); this function automatically determines the areas of the polyhedron facets at any given resolution. The optimization parameters are now the coefficients of the series. Since the number of the parameters to be solved for is not large (typically from, say, 40–100), it is advantageous to use the Levenberg-Marquardt optimization scheme (Press et al., 1992). This very fast and robustly converging approach is especially suitable for obtaining the dynamical parameters \( \mathcal{D} \) (section 7). For a typical lightcurve set (i.e., on the order of 1000 brightness measurements), one minimization run is done in seconds. The complete analysis of a target typically requires a few tens of runs started at different initial values for the rotation and scattering parameters (see sections 6 and 7).

4.2. Albedo Variegation

Obviously \( \mathbf{s} \) in equation (5) may be taken to represent the unknown facet albedo or its product with the facet area (depending on the scattering law, more complicated formulations can be considered as well but they are not relevant to the discussion). Shape parameters can thus be separated from those of albedo only by using suitable constraints.

Simulations and inversion results (Cellino et al., 1989; Kaasalainen et al., 2002b) have shown that significant albedo variegation is seldom required even for drastic lightcurve irregularities, while spacecraft images and simple physical considerations imply that asteroid surfaces are not likely to be covered with extensive high-contrast albedo “paintings.” All the lightcurves of the “test-case” asteroids mentioned in section 2 could be completely explained with the asteroids’ global shapes. The slight albedo markings visible to the eye in the disk-resolved probe images are negligible in the disk-integrated sense. Thus we have good reason to attribute lightcurve variation to shape as much as possible and to invoke albedo variegation only when necessary.

One of the advantages of convex modeling is that the result contains a straightforward indicator of the probable strength of albedo variegation on the asteroid’s surface (Kaasalainen and Torppa, 2001). This quantity can be represented in terms of a “residual area”: The effect of the
asymmetry of albedo variegation on the surface is equivalent to that of a surface patch the size of the residual area. If this area is negligible, the lightcurve features are in all probability caused by the shape.

If the residual area is larger than, say, 1% of the total surface area, we must use the above-mentioned principle of shape domination to obtain a plausible shape/albedo result. This can be given straightforwardly in the form of a regularizing function that, e.g., describes a simple one-spot model over a freely adjustable shape (Kaasalainen et al., 2002b). This spot corresponds to the observed residual area. Another possibility is to use physical \textit{a priori} assumptions. For example, asteroids large enough to be molded mostly by self-gravitation can be expected to be quite spheroidal, if the lightcurve variations are not large. In such cases the shape can be approximated as the best-fit ellipsoid, and the albedo map can then be “painted” on it (an early application of this approach is given for 4 Vesta in Cellino et al., 1987).

4.3. \textbf{Uniqueness and Stability}

An important fact is that the global shape resulting from the separation of albedo asymmetry is close to the correct one as long as the albedo markings on the surface are not abnormally bright and/or extensive. This can best be understood via the convex formulation: The overall shape of a convex body may change very little even if the areas of individual facets change considerably. This property could be called Minkowski stability in recognition of the rather subtle mathematical theorems on convex bodies by Minkowski (e.g., Lamberg and Kaasalainen, 2001, and references therein). It is precisely this property that makes convex inversion so robust; it is also the reason why it is much safer to attribute brightness changes to shape rather than albedo. This stability also means that the inversion result is \textit{not} very sensitive to random noise in observations. In fact, the result is insensitive even to the (realistic) choice of the light-scattering model of the surface (section 6).

The convex shape result is not exactly the convex hull of the body but a shape close to it, “trimmed” such that its shadowing properties best mimic those of the original, often strongly nonconvex shape. The nonconvexities can be seen as deviations from this basic shape. Minkowski stability means that the local curvature may be trimmed in various ways at individual locations on the surface without influencing the global shape of the convex model very much. In addition to providing this stability, only the convex formulation admits uniqueness theorems (Kaasalainen et al., 1992a; Kaasalainen and Torppa, 2001), which is invaluable for the reliability of the inverse solution.

The global characteristics of even quite strongly nonconvex bodies can well be recognized from the convex representation. This is illustrated by Figs. 3 and 4; Fig. 3 portrays the radar-based model of 6489 Golevka viewed at 20° below the equator, while Fig. 4 shows the lightcurve-based convex model at the same geometry. The convex model fitted the observations within the noise, i.e., no nonconvexities were required to explain the lightcurve features. This is quite typical of groundbased observations.

5. \textbf{NONCONVEX MODELS}

Lightcurve observations can usually be well explained with a convex model even when the data are produced by a shape known to have large concave features. Thus we can conclude that lightcurves seldom carry detailed information...
on nonconvex features. This is mostly due to the fact that solar phase angles have to be very large to cause striking shadowing effects. Thus the signature of nonconvex features is usually drowned in the noise at low and intermediate solar phase angles. Another significant factor is that the light-scattering properties of the surface can seldom be modeled very accurately.

From the technical point of view, the main complications in nonconvex modeling are that all uniqueness theorems are lost, and the parameter space is usually peppered with local minima especially if there is noise in the data. A practical approach is to use a short function series for the radii of the vertices of a triangulated surface in given directions. The approach is to use a short function series for the radii of the minima especially if there is noise in the data. A practical lost, and the parameter space is usually peppered with local minima especially if there is noise in the data. A practical lost, and the parameter space is usually peppered with local minima especially if there is noise in the data. A practical

Nonconvex models usually do not really reach lower $\chi^2$'s than convex ones. Minkowski stability no longer applies to them; thus, even in theory, reliable general nonconvex inversion requires highly accurate observations, very favorable observation geometries, and an accurate scattering model. Smoothness regularization is essential, and it is also useful to regularize the rotation axis to align with the axis of the maximum moment of inertia (cf. Pravec et al., 2002). Fortunately, large flat areas on the convex solution already indicate the presence and locations of major nonconvex features.

The convex solution can indeed be seen as the “basic” shape of the object: Various nonconvex perturbations give very similar fits, but the unperturbed shape is stable. We know that asteroids are not convex. However, due to its inherent instability, a nonconvex model can be regarded as reliable only if it fits the data clearly better than a convex one. In other words, the convex formulation provides us with a good diagnostic tool: If we happen to stumble upon one of the very rare objects whose lightcurve data cannot be fitted well with a convex model, we can be quite certain that there must be considerable global concavities on the surface. Contact binaries with a distinctly double-lobed appearance are good candidates for such a class of objects.

6. SCATTERING MODELS AND PHASE CURVES

Accurate, universal few-parameter descriptions of the scattering properties of the surface material are hard to come by. The existing models, such as those by Hapke or Lumme and Bowell (see Bowell et al., 1989), are still inadequate and known to produce ambiguous and unrealistic parameter values in inverse problems. A typical example of a scattering phenomenon not yet properly modeled is the opposition effect, i.e., the brightening near-zero phase angle caused by coherent backscattering and shadowing (e.g., Muinonen et al., 2002).

For lightcurve inversion, the scattering law must be simple; too many parameters and possibilities cause instability and unrealistic results. Also, it is often easier (at least in the first analysis) to simply express the general photometric properties of the surface rather than try to obtain detailed physical parameters. A useful scattering model for this purpose is

$$S(\mu, \mu_0, \alpha) = \left[ f(\alpha) \langle S_{1S}(\mu, \mu_0) \rangle + c S_L(\mu, \mu_0) \right]$$

which combines single (Lommel-Seeliger term $S_{1S}$) and multiple scattering (Lambert term $S_L$) with a weight factor $c$ for the latter. For the sake of convenient inversion, the phase function $f(\alpha)$ is taken to multiply the sum of the single and multiple scattering terms. Such a simplification can be justified by the ambiguity in what should be called single or multiple scattering in a medium consisting of small particles and their aggregates. In this formulation $f(\alpha)$ can be determined afterward from a set of scale factors (obtained by dividing the average observed brightness by the corresponding model brightness) for each lightcurve while it does not have to be known when solving for the other parameters. An exponential and linear model is a versatile choice for this purpose (S. Kaasalainen et al., 2001; Muinonen et al., 2002)

$$f(\alpha) = a \exp \left( -\frac{\alpha}{d} \right) + k\alpha + 1$$

where $a$ and $d$ are the amplitude and scale length of the opposition effect, and $k$ is the overall slope of the phase curve (with the linear part at zero phase angle normalized to unity).

The parameters of any given scattering model can be directly incorporated in the optimization procedure. Hapke, Lumme-Bowell, and the above scattering model all give quite similar results for the shape and the rotational state (M. Kaasalainen et al., 2001). Inversion should obviously be performed with at least two different scattering laws to establish a rough error estimate. Once the scattering characteristics, shape, and the rotation period and pole are known for the asteroid, one can compute “proper” reference phase curves as functions of solar phase angle in, e.g., an equatorial illumination and viewing geometry. Such well-defined phase curves (see Fig. 5), defined as the intensity time average of the model lightcurve as a function of $\alpha$, are naturally more reliable than badly defined curve fits obtained more or less directly from observed magnitudes. In this way one can also define other characteristics such as the amplitude-phase relationship (Zappalà et al., 1990). Reference phase curves are a stable means of representing an object’s global photometric characteristics; as Fig. 5 shows, different scattering laws produce very similar phase curves for 433 Eros. The phase curves from the Hapke model (dotted line) and equations (7) and (8) (solid line) differ slightly only at the
over long periods of time and various observing geometries are used, the only reliable way of estimating the sidereal period is to include it in the general pole and shape analysis.

Let \( \mathbf{r}_{\text{ecl}} \) denote a vector in the ecliptic coordinate frame where the origin is translated to the asteroid. This vector transforms to the vector \( \mathbf{r}_{\text{ast}} \) in the asteroid’s own frame (where z-axis is aligned with the rotation axis) by the rotation sequence

\[
\mathbf{r}_{\text{ast}} = \mathbf{R}_z(\beta) \mathbf{R}_z(90^\circ - \lambda) \mathbf{R}_y(\omega) \mathbf{r}_{\text{ecl}} \tag{9}
\]

where \( t \) is the time and \( \mathbf{R}_z(\alpha) \) is the rotation matrix corresponding to the rotation of the coordinate frame through angle \( \alpha \) in the positive direction about the i-axis. The angle \( \phi_0 \) and the epoch \( t_0 \) can be chosen at will.

The directions \( \mathbf{E}_\text{E} \) and \( \mathbf{E}_\text{S} \) of Earth and the Sun as seen from the asteroid are now simple functions of \( \beta, \lambda, \) and \( \omega \), so these parameters can be included in equations (2) or (3). Obviously there are several local minima in \( \chi^2 \), so multiple initial values for the pole and the period must be applied. The case of the pole is the simplest. A standard choice is to use a few directions in each octant of the celestial sphere as starting points; such a grid will usually cover all the local minima. One can also use the pole estimates of possible previous models. [For methods of obtaining simple first estimates of rotation parameters, see Magnusson et al. (1989, 1996).]

If the lightcurve set covers many years and apparitions, the period space is filled with densely packed local minima. The smallest separation \( \Delta P \) of the local minima when plotting \( \chi^2 \) as a function of the trial period \( P \) is roughly given by

\[
\Delta P = \frac{1}{2} \frac{P^3}{T} \tag{10}
\]

where \( T = \max(t - t_0) \) within the lightcurve set. This derives from the phenomenon that if \( P \) is changed by \( \Delta P \), the minima and maxima of the model lightcurve at \( t_0 \pm T \) will undergo a phase shift of \( 180^\circ \); for the typical double-sinusoidal lightcurves, they will then be at roughly the same places as with \( P \). One should thus use initial periods less than \( \Delta P \) apart from each other covering the whole of the interval within which \( P \) can be expected to lie. Lightcurve inversion uses all apparitions and observations for the period estimate; thus, the more apparitions that are available, the more pronounced the correct local minimum of the period is.

Error estimates for the pole and the period are not completely obvious. Formal errors (such as the square roots of the elements of the covariance matrix) should not be reported; they are worthless in practice since the effects of random noise are usually negligible compared to systematic and model errors. A practical and robust way of estimating the errors is to perform a series of optimizations with different light-scattering models and initial parameter values. The results with \( \chi^2 \) close to the best one describe the parameter distribution; this gives at least a lower bound to the error.

### Fig. 5.

The reference phase curve for 433 Eros obtained with the exponential-linear phase function (8) (solid line), together with the phase curve obtained with the best-fit Hapke parameters (dotted line), and the computed maxima at the equatorial geometry (dashed line). Observed average brightnesses are plotted with asterisks, and the observed maxima with diamonds (note the displacing effect of nonequatorial geometries).

7. ASTEROID ROTATIONS AND PHOTOMETRIC DATA

#### 7.1. Single-Period Lightcurves

The great majority of asteroids are principal-axis rotators exhibiting single-period lightcurves, whereas tumbling bodies and binary systems produce multiple-period lightcurves. An asteroid in the relaxed state of principal-axis rotation revolves about its angular momentum vector \( \mathbf{M} \) coinciding with the asteroid’s axis of maximum moment of inertia, so the rotational state is fully described by three parameters: the ecliptic latitude \( \beta \) and longitude \( \lambda \) of the direction of the rotation axis (pole) and the rotation speed \( \omega \). The sidereal period is \( P = 2\pi/\omega \). The asteroid always rotates in the positive direction about the pole, i.e., \( \omega \) is always positive. Prograde and retrograde motion are indicated by positive and negative \( \beta \) respectively.

A rough estimate of the current synodic period is easily obtained by observing a lightcurve long enough to see a repeated pattern; an improved estimate can be obtained by fitting a Fourier series to a few consecutive lightcurves, if the orbital motion is slow compared to the rotational one. The often-used method of building a composite lightcurve from the data of neighboring observing nights should, however, be practiced only with extreme caution. When data
If the solar phase angle were always exactly zero, it would be impossible to determine the object’s sense of rotation from disk-integrated data since both $M$ and $-M$ would give equally good solutions. Similarly, if the object always moved exactly in the plane of the ecliptic, both $M$ and the reflection of $-M$ in the ecliptic plane (i.e., two $\lambda$s differing by 180°) would be acceptable solutions. Many asteroids often come close to one or both of these limiting cases, resulting in two to four possible solution regions. In these cases it is particularly important to perform a simultaneous solution of both the shape and rotational state, because even if the resulting $x^2$s of the different solutions turn out not to be sufficiently different, the shape solution can still betray the wrong pole by its more unrealistic appearance.

### 7.2. Multiple-Period Lightcurves

Recently, mainly due to the establishment of systematic photometric monitoring programs of NEAs (Mottola et al., 1995; Pravec et al., 1998a; Erikson et al., 2000), the presence of multiple-period lightcurves has been detected. Such observations, when the effects of the relative motion between the target and observer can be neglected or compensated for, can be subjected to standard power-spectrum and multidimensional Fourier analysis (Press et al., 1992). With these methods, the probable underlying frequencies of the data can be singled out and given initial estimates.

A Fourier series representation can also be used for checking the quality of the data and highlighting the possible patterns against noise. In general, the brightness $L$ of the object can be expanded as

$$L = \sum_{i,j} a_{ij} \cos(i\omega_t t + j\omega_s t) + b_{ij} \sin(i\omega_t t + j\omega_s t)$$  \hspace{1cm} (11)

where $\omega_t = 2\pi/P$ and $\omega_s$ are the rotational and spin frequencies, respectively. The presence of binary asteroid systems has long been postulated on the basis of theoretical considerations (e.g., Weidenschilling et al., 1989), and their existence was confirmed at the turn of the millennium with such techniques as photometric observations, adaptive optics, and radar [for the last two, see Merline et al. (2002) and Ostro et al. (2002)]. Photometrically, the clearest indication of a binary system involves two periods. It has been suggested that particular lightcurve features could represent the signature of binary structures where the two bodies are in contact or separated but remain in synchronous rotation with their mutual orbital motion (Van Flandern et al., 1979). However, especially for contact binaries such interpretations can usually be made only indirectly (Kaasalainen et al., 2002a) because of the small photometric information content on nonconvexities as discussed in section 5.

In a binary system, the presence of a companion body causes occultation-eclipse events under certain viewing and illumination geometries. If the central-body rotation is not synchronous with the orbital period of the secondary body, an observer would possibly detect two additive components in the lightcurves (in intensity space), one related to the rotation of the primary and one to the orbital motion of the ordinary lightcurves. The combination of the frequencies is perhaps best understood when one considers the apparent frequency in the limiting case of principal-axis rotation: The two frequencies are still present, but they are only seen as the single frequency $f_\nu + f_\psi$. When the body is slightly tilted, an additional spectrum peak corresponding to the precession frequency will make its appearance.
secondary. Under certain observing conditions, both processing and binary systems can produce lightcurves that are qualitatively similar and whose correct identification requires accurate photometry and good coverage.

The key to the identification of binaries is the additive nature of the two components in the binary system. If we subtract from the lightcurve the periodic component corresponding to the rotation of the primary, and we are left with the unequivocal signature of eclipse-occultation events, we can positively identify the asteroid as a binary system. A number of NEAs have been interpreted as binary systems on the basis of their lightcurves; among the first were 1994 AW1 (Pravec and Hahn, 1997), 3671 Dionysus (Mottola et al., 1997a), and 1991 VH (Pravec et al., 1998b).

Among the best-observed cases of binary systems whose binary nature and dynamical model has been inferred by photometric measurements alone is 1996 FG3 (Mottola and Lahulla, 2000; Pravec et al., 2000). Figure 6a shows a lightcurve of 1996 FG3. The solid line represents a Fourier fit of the rotational lightcurve of the primary. Figure 6b shows the lightcurve of 1996 FG3 with the rotational component of the primary removed. The flat-bottomed minimum corresponds to a total superior eclipse, in which the secondary is totally occulted by the primary and/or its shadow. The deeper minimum corresponds to an inferior eclipse event (the secondary in front of the primary). Its amplitude is larger because, under these observing conditions, both the satellite and its shadow obscure the central body for a certain period of time. The unocculted lightcurve between eclipses displays a gentle curvature. This feature is interpreted as the result of an elongated and synchronous secondary, which produces a lightcurve with the same period as the orbital one and with maximum brightness at maximum satellite elongation.

A major reward of the modeling of binary systems is that it is possible to estimate the average bulk density of the bodies without knowing their absolute sizes. Using the scale-free expressions for the volumes of the model shapes and their sizes vs. orbital separation, density follows directly from Kepler’s third law, whereas the definition of the masses would require the absolute size scale. Knowledge of the density of asteroids is of critical importance for understanding their origin and nature, and the characterization of the orbital motion of a satellite around a central body is the only way to determine this quantity from ground-based measurements.

Obviously, general lightcurve inversion of nonsynchronous binaries is even more complicated and ambiguous than nonconvex inversion. However, the detection of mutual event features in lightcurves places strong constraints on the geometry of the system, and by making reasonable assumptions one can use a simplified approach to infer the basic properties of the system. For example, the assumptions used in the case of 1996 FG3 (Mottola and Lahulla, 2000) are that (1) the central body and its satellite have the same photometric properties, (2) the satellite revolves round the primary in a circular orbit confined to the primary’s equatorial plane, and (3) the secondary rotation is synchronous with its orbital period. These assumptions are introduced to limit the number of free parameters in the minimization problem, while still allowing a reasonable solution. In particular, the second assumption is justified by the timing of the occultation events, which implies a very small deviation from a circular orbit, whereas the third assumption is justified by the observed periodicity of the lightcurve of the secondary body.

Both the lightcurve and the dynamical characteristics of the binary system depend only on the relative sizes and separation of the bodies, so all geometric quantities are left scale-free. In the first analysis (in which lightcurves are not available at several observing geometries), both the central object and its satellite can be treated as triaxial ellipsoids approximated by polyhedra with triangular facets; eclipses and occultations are taken into account by means of ray-tracing techniques. The rotational lightcurve amplitude of the central body constrains its A and B semiaxes, while C is left as a free parameter. In addition, the measurement of

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**Fig. 6.** (a) A lightcurve of the binary asteroid 1996 FG3. (b) The same lightcurve as in (a), but with the rotational component of the binary removed. The occultation lightcurve for the best model solution is shown as a solid line.
the amplitude of the lightcurve of the secondary allows one to express the a and b satellite semiaxes as a function of C. The c satellite semiaxis is not constrained by the lightcurves in this analysis. As its influence on the uncertainty of the density determination is rather small, one can assume $b:c = 1$ in the modeling procedure. In the case of 1996 FG3 the scattering law is the five-parameter Hapke model with average parameters for C-type asteroids. To find the model that best describes the binary system, a grid search is performed over the parameter space (defined by the central body semiaxis C, the orbital radius R, and the ecliptic longitude and latitude $\lambda, \beta$ of the orbital plane normal) to find the solution that minimizes the residuals to the observed lightcurves. The occultation lightcurve for the best-model solution is shown in Fig. 6b as a solid line.

9. CONCLUSIONS AND FUTURE WORK

Lightcurve inversion is an efficient deconvolution technique whose resolving capacity lies, roughly speaking, between space telescope and radar capabilities. The particular advantage of photometric observations is the wide distance range of analyzable objects, extending from near-Earth to main-belt asteroids. The rotational state as well as the characteristic light-scattering properties and large-scale shape features of an object can be well inferred from good-quality data by lightcurve inversion. The global shape of the target can usually be best described with a convex model, as lightcurves seldom carry information on small shape details or nonconvex features.

The solution of the inverse problem is detailed and stable if accurate measurements made at various observing geometries are available; the key principle is thus to conduct well-planned observational campaigns. Well-equipped amateur observers and the development of automatic telescopes should be of considerable assistance in this. Near-Earth asteroids are particularly rewarding targets, since a comprehensive model of an NEA can often be constructed after one suitable apparition, i.e., an observation span of only a few months.

The requirements of various observing geometries are not very stringent. Almost all asteroids to which traditional triaxial ellipsoid methods can be applied can be analyzed with the general method as well. The main objective is to observe the target at as many ecliptic longitudes (and latitudes) and solar phase angles as possible. Thus, a preliminary model of a main-belt asteroid can be built from two suitable apparitions, while three or four apparitions are usually sufficient for the construction of a good model (Kaasalainen et al., 2002b). While accuracy is desired, even substantial random noise in lightcurves does not preclude their use for modeling purposes; also, relative photometry is sufficient for shape and rotation analysis.

There are several aspects of photometric analysis that should be investigated further; we briefly list here but a few. Though no unique theorems can be shown for binary asteroids, dynamical constraints or other types of regularization can be used to provide more detailed solutions. Binaries with more complex rotational states are also possible. Theoretical models of light scattering are still not adequate; neither is it clear how well the physical scattering parameters of the surface can really be determined from disk-integrated photometric data. Such parameters include not only the characteristics of the surface regolith particles, but also statistical topographic variations on small size scales. Lightcurve observations are not restricted to brightness measurements at visual wavelengths. Observations in the IR, together with theoretical models of thermal emission (Harris and Lagerros, 2002), can be particularly valuable in deducing the albedo variegation and composition of the surface.

Perhaps the most interesting future prospect is the use of complementary data simultaneously with photometry in asteroid modeling. Examples of such data include interferometric observations (speckle or other types), timings of stellar occultations, and precise astrometric measurements (from orbiting instruments) for which the photocenter of the target does not coincide with its center of mass. The full Stokes vector of light also includes the state of polarization, but so far not many polarimetric observations of asteroid surfaces have been carried out (Cellino et al., 1999), and theoretical modeling hardly exists. Complementary data will undoubtedly prove to be very valuable in modeling details and removing possible ambiguities.

REFERENCES


