Ward, Colombo and Franklin (1976; hereafter referred to as WCF) have proposed that the anomalously high proper eccentricity and inclination of Mercury result from the passage of time-variable secular resonances with Venus, which were driven by decrease in the solar \( J_2 \) associated with a primordial phase of spin-down. WCF suggested, but did not treat, the possibility that the dissipation of a primordial solar nebula could have led to other secular resonances. The present work treats this suggestion, and proposes that such resonances can account for the proper eccentricities of Mars and of the asteroids.

In treating this problem, a difficulty exists in that the nebular disturbing function, which gives the action of the nebula on a planet or test body, involves divergent integrals. This difficulty has been resolved by introducing the mathematical fiction of a nebula which extends to the time-variable distance \( r_p \) from the Sun to the planet or body, and by restricting consideration to a planar nebula. Following Weidenschilling (1977), the nebula is modelled as an axisymmetric disk with surface density \( \sigma \) given by

\[
\sigma = \sigma_o (a/a_o)^{-s}
\]

where \( a \) = semimajor axis; \( a_o \) = a reference value, taken as 1 AU; \( \sigma_o = 5 \times 10^{-4} \, m_p/AU^2 \) for a total nebular mass of 0.03 \( m_p \) lying between 0.5 and 30 AU. For a nebula extending from \( a_1 \) to \( a_2 \) and a body at \( a_p \), \( a_1 < a_p < a_2 \), define \( a_1 = a_1/a_p, \, a_2 = a_2/a_2 \). The time-averaged nebular disturbing function \( \bar{R}_N \) then is given:

\[
\bar{R}_N = \pi G \sigma_o a_p^3 \int_a^{a_1} a^{-1-s} \left( 1 + \frac{1}{4} (s-1) (s-2) e^2 \right) \left( \int_{a_1}^{a_2} a^{-2-s} \, da + \int_{a_2}^{a_2} a^{-2-s} \, da \right) + \frac{1}{4} \int a^{-2-s} \left( (1-s) b_1^2 (a_1) - a_1^2 b_1^2 (a_1) + a_1^2 b_2^2 (a_1) \right) + \frac{1}{a_1^2} \left( (1-s) b_1^2 (a_2) - a_2^2 b_1^2 (a_2) + a_2^2 b_1^2 (a_2) \right) \right) \, da
\]

\[a_1 = a_1/a_p, \, a_2 = a_2/a_2 \]

The quadrature in eq. (3) has the numerical value 4.377345, denoted \( F \).

With eq. (3), it is possible to treat a planar Solar System consisting of the Sun, Jupiter, Saturn, the nebula, and a massless test body. The origin of inclinations cannot be treated in this problem, but the origin of eccentricities is treated via adaptation of the second-order theory of secular perturbations given in Brouwer and Clemence (1961), chapter XVI. The nebula is considered to retain the form of eq. (1) and to follow a dissipation law: \( \sigma_o/\sigma_p = - \sigma_o \tau_N / \tau_N \); \( \tau_N \) = nebular dissipation time.

Let \( m_J, \, m_S \) be the masses of Jupiter and Saturn; \( n_J, \, n_S \) their mean motions; \( a_J, \, a_S \) their semimajor axes. Define

\[
A_{11} = m_s / \left( 4 \pi n_J \right) \left( \frac{a_J}{a_S} \right) b_{3/2} = - \pi G \sigma_o / \left( 4 a_J \right) \theta_J ; \quad A_{12} = \frac{m_S}{4 \pi n_J n_S b_{3/2} (a_J/a_S) \theta_J} \theta_S
\]

\[
A_{22} = \frac{m_J}{4 \pi n_J n_S b_{3/2} (a_J/a_S) \theta_S} - \pi G \sigma_o / \left( 4 a_S \right) \theta_S \quad A_{21} = - \frac{m_S}{4 \pi n_J n_S b_{3/2} (a_J/a_S) \theta_J} \theta_S
\]

Two eigenfrequencies contribute to the perihelion advance rates of Jupiter and Saturn:

\[
s_1, \, s_2 = \frac{1}{2} \left( (A_{11} + A_{22}) + \left( (A_{11} - A_{22})^2 + 4 A_{12} A_{21} \right)^{1/2} \right)
\]
ORIGIN OF ASTEROIDAL ECCENTRICITIES

T. A. Heppenheimer

The test body has perihelion advance rate $g$:

$$g = \frac{1}{4} n_0 a_p^2 \lambda_{1/3} \left[ (m_{1}/a_1) h_{1}^{(1)} + (m_{2}/a_2) h_{2}^{(1)} \right] - \frac{\pi F}{4a_p} \sigma_o$$

At any $a_p$, a secular resonance exists if $g = s_1$ or $g = s_2$. Associated with these eigenfrequencies are eccentricity amplitudes; thus, for example,

$$h_j = e_{1j} \sin(s_1 t + \Theta_1) + e_{2j} \sin(s_2 t + \Theta_2); k_j = e_{1j} \cos(s_1 t + \Theta_1) + e_{2j} \cos(s_2 t + \Theta_2)$$

where $\Theta_1$, $\Theta_2$ are phase angles. Then $e_{1j}^2 = h_j^2 + k_j^2$ and $e_S$ is treated similarly. The ratios $e_{1j}/e_{S1} = \lambda_{12}/(s_1 - \lambda_1)$, $e_{2j}/e_{S2} = \lambda_{21}/(s_2 - \lambda_2)$ allow the solution to be made dependent only on $e_{S1}$, $e_{S2}$, present-day values of which are known: $e_{S2} = 0.048188$, $e_{S1} = 0.048628$ (Brouwer and van Woerkom, 1950; hereafter referred to as BVW). Then, according to WCF, the passage of the $s_3$ resonance at $a = a_p$ excites a maximum eccentricity $e_{pj}$, starting from zero:

$$e_{pj} = \frac{1}{4} n_0 a_p^2 \lambda_{1/3} \left[ (m_{1}/a_1) h_{1}^{(2)} + (m_{2}/a_2) h_{2}^{(2)} \right] \left( \frac{\pi F}{d(q-s_j)} \right)^{1/2}$$

With the foregoing, eq. (7) leads to definition of time scales $T_1$, $T_2$ as those scales for dissipation of $e_0$ leading respectively to $e_p = e_{S1}$ or $e_p = e_{S2}$. Then $e_{p1} = e_{S1}(T_p/T_1)^{1/4}$, $e_{p2} = e_{S2}(T_p/T_2)^{1/4}$.

With this model, there are the results:

Behaviour of the secular resonances: At $e_0 = 5 \times 10^{-4}$ $\text{m}_0/\text{AU}^2$, prior to the onset of nebular dissipation, the $s_2$ resonance is at 3.679 AU, the $s_1$ at 4.050. As $e_0 \rightarrow 0$, $s_2 \rightarrow 0.623$ AU, $s_1 \rightarrow 1.846$ and the curves of $a_p(s_1(s_2))$ behave approximately as $\ln^{-2} e_0$. The $s_1$ asymptote compares with the location of the corresponding secular resonance in BVW's eight-planet solar system, 2.03 AU. Hence only the $s_2$ resonance can influence Mars, Earth and Venus.

Eccentricity of Mars: In the BVW solution, Mars has proper eccentricity $e_M = 0.0723712$. This can be excited with the BVW $e_{S2}$ at $e_0 = 6 \times 10^{-6}$ $\text{m}_0/\text{AU}^2$, with $T_p = 30,000$ years. The associated $T_2$ is 13,500 years.

Eccentricity of Earth and Venus: As the $s_2$ resonance penetrates Sunward of Mars, $T_2$ increases markedly: to 41,000 years at Earth, 57,000 years at Venus. Hence if $T_p$ remains at 30,000 years (or decreases further) as the nebular dissipation nears completion, there will be no unacceptably large values of eccentricity for Earth or Venus.

Distribution of asteroidal eccentricities: If the asteroids are acted upon by two resonances, exciting amplitudes $e_{p1}$ and $e_{p2}$, the final $e_p$ is

$$e_p = \left( e_{p1}^2 + e_{p2}^2 - 2e_{p1}e_{p2} \cos u \right)^{1/2}$$

where $u$ is a random phase angle. Hence the probability distribution of $e_p$ has peaks (singularities) at $e_p = e_{p2} + e_{p1}$. This conflicts with the observed distribution of asteroidal eccentricities, which is approximately Gaussian with mean value $e_p \approx 0.16$. One resolves this difficulty by positing the existence of other secular resonances. An effective means for introducing such resonances is a nonuniform nebular evolution which produces a dense gaseous disk in the inner Solar System. Formation of this disk can cause the $s_1$ and $s_2$ resonances to advance inward across the asteroid zone; incorporation of the disk into the Sun can cause them to retreat outward again. The subsequent overall nebular dissipation then produces the irreversible inward advance. Such multiple resonances have the effect of introducing additional amplitudes and random phase angles into eq. (8); with multiple random variables, the $e_p$ distribution shows regression to the mean and can readily be approximately Gaussian.

Excitation of asteroidal eccentricities: While (for example) $s_2$ was pump-
ING ON MARS, $s_1$ WAS SCATTERING ASTEROIDS AT $a_p = 2.15$ AU. IN GENERAL, THE PROPOSED SEQUENCE IS THAT FIRST THE $s_2$ RESONANCE EXCITED ASTEROIDAL $e_p$'S TO A UNIFORM MEAN VALUE OF 0.16, THEN THE $s_1$ SCATTERED $e_p$ ABOUT THIS VALUE AS IN EQ. (8). WITH THE BVW VALUES OF $e_{s1}$, $e_{j2}$, THEN AT $a_p = 2.15$ AU, $e_{p2} = 0.061$. THIS IS ACCEPTABLE SINCE ADDITIONAL SECULAR RESONANCES COULD HAVE CONTRIBUTED THE REMAINING $\sim 0.08$ OF SCATTER IN $e_p$.

SO FOR THE LATTER STAGES OF NEBULAR DISSIPATION $(\omega \approx 2.5 \times 10^{-6}$ m$/AU^2)$, USE OF THE BVW DATA INVOLVES NO EVIDENT CONTRADICTION OR INCONSISTENCY.

PRIMORDIAL BEHAVIOR OF $e_{s1}$, $e_{j2}$, $\tau_n$: FOR $e_o = 2.3 \times 10^{-5}$ m$/AU^2$, THE $s_2$ RESONANCE IS AT $a_p = 2.2$ AU, THE INNER EDGE OF THE ASTEROIDS. IF $e_{p1}/e_{p2}$ FOR THIS $e_o$ THEN USE OF THE BVW VALUES FOR $e_{s1}$, $e_{j2}$ STILL IS APPROPRIATE AND $\tau_n \approx 41,000$ YEARS. BUT FOR LARGER $e_o$, USE OF THE BVW VALUES CANNOT BE SUSTAINED SINCE FOR $e_{p1}/e_{p2} \sim 1/2$, $e_{s1}/e_{j2}$ MUST BE AS LARGE AS 4. HENCE IT IS PLausible THAT INITIALLY SATURN AND JUPITER HAD NOTICEABLY DIFFERENT ECCENTRICITY AMPLITUDES. ANY ASSUMPTION AS TO THEIR VALUES IS SPECULATIVE; BUT IF $e_{j2} \sim 0.01$ THEN $\tau_n \sim 100,000$ YEARS AT $e_o = 2 \times 10^{-4}$ m$/AU^2$, AT WHICH VALUE THE $s_2$ RESONANCE WAS AT 3.3 AU, THE OUTER LIMIT OF MAIN-BELT ASTEROIDS. THUS $\tau_n$ MAY HAVE STAYED IN A NARROW RANGE, 30,000 TO 100,000 YEARS, THROUGHOUT VIRTUALLY THE WHOLE OF THE NEBULAR DISSIPATION. THIS COMPARES WITH AN UPPER BOUND OF $10^6$ YEARS FOUND BY WCF FOR WHAT MAY HAVE BEEN A CLOSELY RELATED PROCESS, SOLAR SPIN-DOWN.

ASTEROIDAL INCLINATIONS: WHILE THE PRESENT THEORY DOES NOT TREAT INCLINATIONS, THERE IS THE OBVIOUS POSSIBILITY THAT THEY AROSE VIA SECULAR RESONANCES WITH EIGENFREQUENCIES CONTRIBUTING TO NODAL REGRESSION RATES FOR JUPITER AND SATURN. SUCH AN EXPLANATION COULD ACCOUNT FOR A FEATURE WHICH OTHERWISE IS DIFFICULT TO EXPLAIN: ASTEROIDAL VALUES OF $\sin i_p$ are systematically higher than values of $e_p$, on average. If asteroidal elements were produced by a random-scattering process (e.g. gravitational scattering off massive bodies which now are part of Jupiter), $\sin i_p$ would be higher than $e_p$, on average.

ORIGIN OF THE KIRKWOOD GAPS: THE THEOREMS OF LAPLACE, POISSON AND MESSAGE (1976) GUARANTEE INVARIANCE OF $e_p$ UNDER SECULAR PERTURBATIONS. HENCE THE PRESENT THEORY IS CONSISTENT WITH A THEORY FOR ORIGIN OF THE ASTEROIDAL KIRKWOOD GAPS (HEPPENHEIMER, 1978): THAT THEY ARE PRIMORDIAL, REPRESENTING REGIONS WHERE ASTEROIDS FAILED TO FORM BY ACCRETION OF PLANETESIMALS. SUCH PRIMORDIAL GAPS THEN WOULD NOT BE FILLED IN BY SUBSEQUENT PROCESSES WHICH PRODUCED THE ASTEROIDS' VALUES OF $e_p$ AND $i_p$, IF THE PRESENT THEORY IS CORRECT. THERE IS NO REQUIREMENT THAT THE GAPS FORMED AFTER THE ORIGIN OF ASTEROIDS HAD GONE TO COMPLETION.

CONCLUDING COMMENTS: THE RESULTS OF THIS WORK INDICATE THAT THERE ARE PROMISING RESULTS TO BE OBTAINED FROM STUDY OF THE GRAVITATIONAL EFFECTS OF A PRIMORDIAL SOLAR NEBULA, AND THAT MUCH DEeper STUDY IS WARRANTED.

REFERENCES