SECULAR RESONANCES AND THE ORIGIN OF ECCENTRICITIES OF MARS AND THE ASTEROIDS. T. A. Heppenheimer, Center for Space Science, 11040 Blue Allium Ave., Fountain Valley, California 92708

Ward, Colombo and Franklin (1976; hereafter referred to as WCF) have proposed that the anomalously high proper eccentricity and inclination of Mercury result from the passage of time-variable secular resonances with Venus, which were driven by decrease in the solar J_2 associated with a primordial phase of spin-down. WCF suggested, but did not treat, the possibility that the dissipation of a primordial solar nebula could have led to other secular resonances. The present work treats this suggestion, and proposes that such resonances can account for the proper eccentricities of Mars and of the asteroids.

In treating this problem, a difficulty exists in that the nebular disturbing function, which gives the action of the nebula on a planet or test body, involves divergent integrals. This difficulty has been resolved by introducing the mathematical fiction of a nebula which extends to the time-variable distance r_p from the Sun to the planet or body, and by restricting consideration to a planar nebula. Following Weidenschilling (1977), the nebula is modelled as an axisymmetric disk with surface density σ given by

$$\sigma = \sigma_o (a/a_o)^{-s} \tag{1}$$

a = semimajor axis; a_o = a reference value, taken as 1 AU; $\sigma_o = 5 \times 10^{-4}\ m_\odot/AU^2$ for a total nebular mass of 0.03 $m_\odot$ lying between 0.5 and 30 AU. For a nebula extending from a_1 to a_2 and a body at a_p, $a_1 < a_p < a_2$, define $\alpha_1 = a_1/a_p$, $\alpha_2 = a_p/a_2$. The time-averaged nebular disturbing function $\overline{R_N}$ then is given:

$$\begin{aligned}\overline{R_N} = {} & \pi G \sigma_o a_o^{\,s} a_p^{\,1-s}\left(1 + \tfrac{1}{4}(s-1)(s-2)e^2\right)\left(\int_{\alpha_1}^{1} \alpha^{1-s} b_{1/2}^{(0)}\, d\alpha + \int_{\alpha_2}^{1} \alpha^{s-2} b_{1/2}^{(0)}\, d\alpha\right) \\ & + \tfrac{1}{4} e^2 \{\alpha_1^{\,2-s}\left((1-s) b_{1/2}^{(0)}(\alpha_1) - \alpha_1 b_{3/2}^{(1)}(\alpha_1) + \alpha_1^{\,2} b_{3/2}^{(0)}(\alpha_1)\right) \\ & \quad + \alpha_2^{\,s-1}\left((s-2) b_{1/2}^{(0)}(\alpha_2) - \alpha_2 b_{3/2}^{(1)}(\alpha_2) + \alpha_2^{\,2} b_{3/2}^{(0)}(\alpha_2)\right)\} + O(e^4)\end{aligned} \tag{2}$$

G = constant of gravitation; e is eccentricity; $b_k^{(i)}$ are Laplace coefficients. Following Weidenschilling (1977), we take $s = 3/2$. Also we take $a_1 = 0$, $a_2 = \infty$, and eq. (2) becomes

$$\overline{R_N} = 2\pi G \sigma_o a_o^{\,3/2} a_p^{\,-1/2}\left(1 - \frac{1}{16} e^2\right)\int_0^1 \alpha^{-1/2} b_{1/2}^{(0)}(\alpha)\, d\alpha \tag{3}$$

The quadrature in eq. (3) has the numerical value 4.377345, denoted F.

With eq. (3), it is possible to treat a planar Solar System consisting of the Sun, Jupiter, Saturn, the nebula, and a massless test body. The origin of inclinations cannot be treated in this problem, but the origin of eccentricities is treated via adaptation of the second-order theory of secular perturbations given in Brouwer and Clemence (1961), chapter XVI. The nebula is considered to retain the form of eq. (1) and to follow a dissipation law: $d\sigma_o/dt = -\sigma_o/\tau_N$; τ_N = nebular dissipation time.

Let m_J, m_S be the masses of Jupiter and Saturn; n_J, n_S their mean motions; a_J, a_S their semimajor axes. Define

$$A_{11} = m_s/(4 n_J a_J a_S) b_{3/2}^{(1)}(a_J/a_S) - \pi F \sigma_o/(4 a_J)\ ; \quad A_{12} = \frac{-m_S}{4 n_J a_J a_S} b_{3/2}^{(2)}(a_J/a_S)$$

$$A_{22} = \tfrac{1}{4} m_J a_J n_S b_{3/2}^{(1)}(a_J/a_S) - \pi F \sigma_o/(4 a_S) \qquad A_{21} = -\tfrac{1}{4} m_J a_J n_S b_{3/2}^{(2)}(a_J/a_S)$$

Two eigenfrequencies contribute to the perihelion advance rates of Jupiter and Saturn:

$$s_1,\ s_2 = \tfrac{1}{2}\{(A_{11} + A_{22}) \pm \left((A_{11} - A_{22})^2 + 4 A_{12} A_{21}\right)^{1/2}\} \tag{4}$$

ORIGIN OF ASTEROIDAL ECCENTRICITIES

T. A. Heppenheimer

The test body has perihelion advance rate g:

$$g = \frac{1}{4} n_p a_p^2 \left((m_J/a_J) b_{3/2}^{(1)}(a_p/a_J) + (m_S/a_S) b_{3/2}^{(1)}(a_p/a_S) \right) - \frac{\pi F}{4 a_p} \sigma_o \qquad (5)$$

At any a_p, a secular resonance exists if $g = s_1$ or $g = s_2$. Associated with these eigenfrequencies are eccentricity amplitudes; thus, for example,

$$h_J = e_{J1} \sin(s_1 t + \theta_1) + e_{J2} \sin(s_2 t + \theta_2); \; k_J = e_{J1} \cos(s_1 t + \theta_1) + e_{J2} \cos(s_2 t + \theta_2) \qquad (6)$$

where θ_1, θ_2 are phase angles. Then $e_J^2 = h_J^2 + k_J^2$ and e_S is treated similarly. The ratios $e_{J1}/e_{S1} = A_{12}/(s_1 - A_{11})$, $e_{S2}/e_{J2} = A_{21}/(s_2 - A_{22})$ allow the solution to be made dependent only on e_{S1}, e_{J2}, present-day values of which are known: $e_{J2} = 0.048188$, $e_{S1} = 0.048628$ (Brouwer and van Woerkom, 1950; hereafter referred to as BVW). Then, according to WCF, the passage of the s_j resonance at $a = a_p$ excites a maximum eccentricity e_{pj}, starting from zero:

$$e_{pj} = \frac{1}{4} n_p a_p^2 \left(\frac{m_J}{a_J} e_{Jj} b_{1/2}^{(2)}(a_p/a_J) + \frac{m_S}{a_S} e_{Sj} b_{3/2}^{(2)}(a_p/a_S) \right) \left(\frac{2}{d(g-s_j)/dt} \right)^{\frac{1}{2}} \qquad (7)$$

With the foregoing, eq. (7) leads to definition of time scales τ_1, τ_2 as those scales for dissipation of σ_o leading respectively to $e_p = e_{S1}$ or $e_p = e_{J2}$. Then $e_{p1} = e_{S1}(\tau_N/\tau_1)^{\frac{1}{2}}$, $e_{p2} = e_{J2}(\tau_N/\tau_2)^{\frac{1}{2}}$.

With this model, there are the results:

Behaviour of the secular resonances: At $\sigma_o = 5 \times 10^{-4}$ $m_\odot/AU^2$, prior to the onset of nebular dissipation, the s_2 resonance is at 3.679 AU, the s_1 at 4.050. As $\sigma_o \to 0$, $s_2 \to 0.623$ AU, $s_1 \to 1.846$ and the curves of $a_p(s_j(\sigma_o))$ behave approximately as $\ell n^{-2} \sigma_o$. The s_1 asymptote compares with the location of the corresponding secular resonance in BVW's eight-planet solar system, 2.03 AU. Hence only the s_2 resonance can influence Mars, Earth and Venus.

Eccentricity of Mars: In the BVW solution, Mars has proper eccentricity $e_M = 0.0723712$. This can be excited with the BVW e_{J2} at $\sigma_o = 6 \times 10^{-6}$ $m_\odot/AU^2$, with $\tau_N = 30{,}000$ years. The associated τ_2 is 13,500 years.

Eccentricity of Earth and Venus: As the s_2 resonance penetrates Sunward of Mars, τ_2 increases markedly: to 41,000 years at Earth, 57,000 years at Venus. Hence if τ_N remains at 30,000 years (or decreases further) as the nebular dissipation nears completion, there will be no unacceptably large values of eccentricity for Earth or Venus.

Distribution of asteroidal eccentricities: If the asteroids are acted upon by two resonances, exciting amplitudes e_{p1} and e_{p2}, the final e_p is

$$e_p = \left(e_{p1}^2 + e_{p2}^2 - 2 e_{p1} e_{p2} \cos u \right)^{\frac{1}{2}} \qquad (8)$$

where u is a random phase angle. Hence the probability distribution of e_p has peaks (singularities) at $e_p = e_{p2} \pm e_{p1}$. This conflicts with the observed distribution of asteroidal eccentricities, which is approximately Gaussian with mean value $e_p \sim 0.16$. One resolves this difficulty by positing the existence of other secular resonances. An effective means for introducing such resonances is a nonuniform nebular evolution which produces a dense gaseous disk in the inner Solar System. Formation of this disk can cause the s_1 and s_2 resonances to advance inward across the asteroid zone; incorporation of the disk into the Sun can cause them to retreat outward again. The subsequent overall nebular dissipation then produces the irreversible inward advance. Such multiple resonances have the effect of introducing additional amplitudes and random phase angles into eq. (8); with multiple random variables, the e_p distribution shows regression to the mean and can readily be approximately Gaussian.

Excitation of asteroidal eccentricities: While (for example) s_2 was pump-

ing on Mars, s_1 was scattering asteroids at a_p = 2.15 AU. In general, the proposed sequence is that first the s_2 resonance excited asteroidal e_p's to a uniform mean value of 0.16, then the s_1 scattered e_p about this value as in eq. (8). With the BVW values of e_{S1}, e_{J2}, then at a_p = 2.15 AU, e_{p2} = 0.061. This is acceptable since additional secular resonances could have contributed the remaining $\sim$ 0.08 of scatter in e_p. So for the latter stages of nebular dissipation ($\sigma_o \lesssim 2.5 \times 10^{-5}$ $m_\odot/AU^2$), use of the BVW data involves no evident contradiction or inconsistency.

Primordial behaviour of e_{S1}, e_{J2}, τ_N: For $\sigma_o = 2.3 \times 10^{-5}$ $m_\odot/AU^2$, the s_2 resonance is at a_p = 2.2 AU, the inner edge of the asteroids. If e_{p1}/e_{p2} for this σ_o then use of the BVW values for e_{S1}, e_{J2} still is appropriate and τ_N = 41,000 years. But for larger σ_o, use of the BVW values cannot be sustained since for $e_{p1}/e_{p2} \sim 1/2$, e_{S1}/e_{J2} must be as large as 4. Hence it is plausible that initially Saturn and Jupiter had noticeably different eccentricity amplitudes. Any assumption as to their values is speculative; but if $e_{J2} \sim 0.01$ then $\tau_N \sim$ 100,000 years at $\sigma_o = 2 \times 10^{-4}$ $m_\odot/AU^2$, at which value the s_2 resonance was at 3.3 AU, the outer limit of main-belt asteroids. Thus τ_N may have stayed in a narrow range, 30,000 to 100,000 years, throughout virtually the whole of the nebular dissipation. This compares with an upper bound of 10^6 years found by WCF for what may have been a closely related process, solar spin-down.

Asteroidal inclinations: While the present theory does not treat inclinations, there is the obvious possibility that they arose via secular resonances with eigenfrequencies contributing to nodal regression rates for Jupiter and Saturn. Such an explanation could account for a feature which otherwise is difficult to explain: asteroidal values of sin i_p are systematically higher than values of e_p, on average. If asteroidal elements were produced by a random-scattering process (e.g. gravitational scattering off massive bodies which now are part of Jupiter), sin i_p would be higher than e_p, on average.

Origin of the Kirkwood gaps: The theorems of Laplace, Poisson and Message (1976) guarantee invariance of a_p under secular perturbations. Hence the present theory is consistent with a theory for origin of the asteroidal Kirkwood gaps (Heppenheimer, 1978): that they are primordial, representing regions where asteroids failed to form by accretion of planetesimals. Such primordial gaps then would not be filled in by subsequent processes which produced the asteroids' values of e_p and i_p, if the present theory is correct. There is no requirement that the gaps formed after the origin of asteroids had gone to completion.

Concluding comments: The results of this work indicate that there are promising results to be obtained from study of the gravitational effects of a primordial solar nebula, and that much deeper study is warranted.

REFERENCES

Brouwer, D. and Clemence, G.M. (1961). Methods of Celestial Mechanics. Academic Press, New York

Brouwer, D. and van Woerkom, A.J.A. (1950). Astron. Papers Amer. Ephem. Naut. Almanac 13, Part 2, 79-107

Heppenheimer, T.A. (1978). Astronomy and Astrophysics, in press

Message, J. (1976). Long Time Predictions in Dynamics (V. Szebehely and B. Tapley, eds.), Reidel, Dordrecht (Holland), pp. 279-293

Ward, W.R., Colombo, G., and Franklin, F.A. (1976). Icarus 28, 441-452

Weidenschilling, S.J. (1977). Astrophys. and Space Sci. 51, 153-158