An important process affecting the accretion of a swarm of planetesimals is the stirring of random velocities (orbital $e$ and $i$) by gravitational scattering due to differential rotation of the swarm (Keplerian shear). Greenberg et al. (1978) described a heuristic stirring model in which relative velocities in encounters were determined entirely by the differences in the semi-major axes of the planetesimal orbits, i.e., Keplerian shear dominated. This approach is criticized by Stewart and Wetherill (1988), who point out that the random component of velocity dominates in close encounters, and derive a different expression for the stirring rate from a Boltzmann equation. The purpose of the present work is to examine the applicability and limitations of each approach.

The assumptions of Stewart and Wetherill that random velocities dominate, and that encounters more distant than the scale height of the swarm are neglected, imply that the bodies considered have crossing orbits. Their modeling of the swarm as a continuum, with velocity components defining a pressure tensor, implies that a typical planetesimal can encounter many other bodies in all parts of its orbit. These assumptions may break down for stirring by the largest bodies in the swarm if they are few in number, with mean orbital spacing greater than the swarm thickness. Because the assumption that Keplerian shear dominates implies that orbits do not cross, a stirring model with that basis may be preferable for calculating the effect of the largest bodies.

Stewart and Wetherill derive four expressions for the rate of velocity change due to (A) "viscous stirring" by gravitational scattering, (B) excitation of velocities by collisions, (C) damping by collisions, and (D) energy exchange between bodies of different sizes, tending toward equipartition ("dynamical friction"). Processes B and C are treated numerically by Greenberg et al., while D has no explicit analog in their model. The heuristic stirring model in their 1978 paper has a different functional dependence on planetesimal mass and velocity from the A term of Stewart and Wetherill. The reason for this difference is not clear, because the derivation is not described in detail. A later version (unpublished) has $m$ and $V$ dependences that match those of Stewart and Wetherill. Greenberg et al. used two-body formalism to evaluate angular deflections in distant encounters, which are fundamentally three-body in nature. Here I present results of a new calculation based on the conjunction of two orbiting bodies and compare these results with the effects of stirring by close encounters. For simplicity, both bodies are assumed to have the same mass, $m$.

Assume a test body is at the origin of rotating Hill coordinates. A field body in a non-crossing orbit with separation $\Delta a$ ($e$, $i$, $<\Delta a/a$) has a transverse velocity due to Keplerian shear of $3\Delta a^2/2$ relative to the test body (this is three times the relative velocity assumed by Greenberg et al.; note that there would still be shear in the swarm if $\partial V_K/\partial a = 0$).

The integrated radial and vertical components of the gravity of the field body as it passes through conjunction result in velocity perturbations $\delta V_r \approx 4Gm/30(\Delta a^2 + \Delta z^2)$, $\delta V_z \approx (\Delta z / \Delta a) \delta V_r$. The changes in $e$ and $i$ are...
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\[ \delta v / \sqrt{K} \text{, but because the effects of small impulses on the orbital} \]

elements depend on true anomaly, \( \delta(e^2) = (\delta e)^2 / 2; \delta(i^2) = (\delta i)^2 / 2 \). The rate of encounters at radial separation \( \Delta a \) is \( 3\pi a \Delta n / 2 \), where \( n \) is the number density of field bodies. Integrating through the thickness of the swarm and radially inward and outward from the test body's orbit, we find

\[
\frac{d}{dt} \frac{(e^2 + i^2) v^2}{K} = \frac{d}{dt} (V^2) \approx \frac{8G^2 m^2 n}{3\Omega} \int_{-D_1}^{D_1} \int_{-D_0}^{D_0} \frac{x+z^2 / \chi}{(x^2+z^2)^2} \, dz \, dx
\]

where \( H = V/\Omega \) is the scale height of the swarm, \( \Sigma = 2nmH \) is the surface density, and \( D_0 \), \( D_1 \) are lower and upper limits of \( \Delta a \). Unlike Chandrasekhar's (1960) expression for stellar velocity perturbations, this expression is bounded as \( D_1 \to \infty \), but diverges for \( D_0 \to 0 \). Greenberg et al. performed an analogous integration through the swarm, but did not give the assumed limits. Taking \( D_0 = H \), \( D_1 = \infty \) gives \( \vartheta(V^2)/\vartheta t \approx 1.1G^2 m^2 \Sigma V^2 / \chi \).

For bodies of equal mass, the viscous stirring term of Stewart and Wetherill reduces to \( \vartheta(V^2)/\vartheta t \approx (2.25G^2 m^2 \Sigma V^2 / \chi) \ln \Lambda \), where \( \Lambda \) is a function of the minimum and maximum deflection angles of the relative velocity vector in gravitational scattering events. If we define \( \Theta = Gm/rV^2 \), where \( r \) is the planetesimal radius, one can show that \( \Lambda = [(1+4\rho/3\Omega^2 \Theta^3)/(1+4(1+\Theta)/\Theta^2)]^{1/2} \). For planetesimal density \( \rho = 3g/cm^3 \) at \( a = 1AU \), \( \ln \Lambda \) ranges from 15.6 at \( \Theta = 1 \) to 6.3 at \( \Theta = 10^3 \). Thus, distant encounters contribute a modest (<1%) fraction of the total gravitational stirring of a swarm of uniform bodies. Their effect on a swarm containing a wide range of sizes deserves further investigation.