HORSESHOE ORBIT DRAG. Wm. R. Ward, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109.

A ring of particles occupying a horseshoe orbit region interacts gravitationally with and, in general, exerts a net cumulative torque on a perturbing body. Quinn and Goodman (1) have provided an estimate of this torque in connection with their discussion of sinking satellite galaxies. Their torque expression is proportional to the local gradient of the disk's surface density, $d\sigma/dr$. However, the disturbing potential in the horseshoe region of a perturber in a circular orbit is dominated by a series of overlapping corotation resonances (2) and torques associated with such resonances are known to be proportional to the gradient, $d(\sigma/B)/dr$; where $B = (2r)^{-1}d(r^2\Omega)/dr$ is the Oort constant measuring disk vorticity (2,3). Here we show that this is indeed the case for the horseshoe ring torque as well, clarifying its connection with corotation resonances. In cases where the density gradient is mild, this additional dependence may reverse the expected sign of the torque. Such situations are common in astrophysical applications.

Consider a perturber, $M_p$, in a circular orbit, $r_p$, of mean motion, $\Omega_p$, that is approached from the rear by an interior particle ($m \ll M_p$) in a circular orbit of radius $r_i < r_p$ and mean motion $\Omega_i > \Omega_p$, [assuming the Oort constant $A = (r/2)d\Omega/dr < 0$]. The particle experiences a torque during its approach that increases its angular momentum. This, in turn, promotes the particle to a higher orbit. If the initial differential semi-major axis is small enough, i.e., $|r_p - r_i| \lesssim W \approx |GM_p/2AB|^{1/3}$, the orbit radius will drift above $r_p$ before encounter (1). The relative mean motion $\Omega - \Omega_p$ reverses sign and the particle falls behind the perturber on an outer orbit $r_o > r_p$, having executed the hair-pin curve that constitutes one end of a horseshoe trajectory. Adiabatic invariance guarantees that epicyclic motion is not generated and that the post-encounter orbit is also circular (4). On the other hand, particles initially at $r_o$, approach from the front (as seen in a frame rotating with the perturber) and suffer the opposite fate. They execute a hair-pin curve in front of the perturber and eventually recede in the forward direction along $r_i < r_p$ (5). Hence, the horseshoe orbit has the effect of exchanging inner and outer particles in the ring.

The increase in specific angular momentum from $r_i$ to $r_o$ is $\Delta h = r_o^2\Omega_o - r_i^2\Omega_i \approx 2rB(r_o - r_i)$. Let $\delta m_k = \sigma_k r_k \delta\theta_k \delta r_k$, $k = (i,o)$ be mass elements of the ring that are exchanged by a given horseshoe orbit in an interval $\delta t$, the subscripts denoting their points of origin. There is a net change in angular momentum of the ring if these mass elements are unequal; $\Delta H = (\delta m_i - \delta m_o)\Delta h$. This change must be absorbed by the perturber and is the source of its reaction torque. A density gradient will contribute to this (1), but there is also a contribution from a mapping between unequal differential areas, $r_k\delta\theta_k\delta r_k$. The angular distance traversed in time $\delta t$ is $\delta\theta_k = |\Omega_k - \Omega_p|\delta t$, so that $r_i\delta\theta_i/r_o\delta\theta_o = r_i|\Omega_i - \Omega_p|/r_o|\Omega_o - \Omega_p|$. To compare radial widths, we employ the Jacobi constant, $J = \Omega_p h - E$. Its increment across the mass elements is $|\delta J| = \delta r_k |dJ/dr|_k$. Far from encounter the Jacobi takes the form

$$J = \Omega_p(r^2\Omega) - \frac{1}{2}(r\Omega)^2 - \Phi \tag{1}$$

where $\Phi$ is the background potential determining the unperturbed orbital velocities; $(r\Omega)^2 = rd\Phi/dr$. Differentiating eqn (1) yields,

$$\left(\frac{dJ}{dr}\right)_k = 2r_k B_k(\Omega_p - \Omega_k) \tag{2}$$

implying that $\delta r_i/\delta r_o = r_o B_o|\Omega_o - \Omega_p|/r_i B_i|\Omega_i - \Omega_p|$. Consequently, an exterior area, $r_o\delta\theta_o\delta r_o$, is mapped onto an interior area

$$r_i\delta\theta_i\delta r_i = r_o\delta\theta_o\delta r_o\left(\frac{B_o}{B_i}\right) \tag{3}$$

and the angular momentum received by the perturber due to the exchange is

$$\Delta H = B_o r_o \delta\theta_o \delta r_o(\sigma_o/B_o - \sigma_i/B_i)\Delta h \tag{4}$$

The torque from a given horseshoe orbit is thus $\delta T = \Delta H/\delta t$, i.e.,

$$\delta T \approx 16|A|B^2(r_o - r_p)^3 r\, \delta r\, \frac{d}{dr}\left(\frac{\sigma}{B}\right) \tag{5}$$

where to lowest order, $|\Omega_o - \Omega_p| \sim 2|A|(r_o - r_p)/r$, $(\sigma_o/B_o - \sigma_i/B_i) \sim (r_o - r_i)d(\sigma/B)/dr$, $r_o - r_i \sim 2(r_o - r_p)$, and all derivatives are evaluated at $r_p$. The total torque is now found by integrating eqn (5) out to the halfwidth of the ring, $r_o - r_p = w \lesssim W$:

$$T \approx 4\sigma|A|Bw^4\left(\frac{d\,\ell n(\sigma/B)}{d\,\ell n\,r}\right) \tag{6}$$

For a Keplerian disk $[A = -3\Omega/4,\ B = \Omega/4]$ with a surface density $\sigma \propto r^{-s}$, eqn (6) reads, $T = 3/4(3/2-s)\sigma w^4\Omega^2$.

The horseshoe orbit torque produces an effective drag on the perturber that contributes to its orbital drift, $\dot{r} = (\Sigma T)/2rBM_p$, where $\Sigma T$ is the sum of all torques exerted on the perturber. If the object is embedded in a disk, these torques include those associated with Lindblad resonances in both interior and exterior portions of the disk (1,2,6). Unlike Lindblad resonances, there is no wave response at a corotation resonance that can transport angular momentum away from the resonance zone. Particles are set into libration, which returns them to their original orbital position in a time $2P \sim 4\pi/|\Omega_k - \Omega_p| \sim 2\pi r/|A(r_k - r_p)|$. As a result, the zone can saturate, shutting off the torque (1). The outermost horseshoe orbits, which contribute most of the torque, have the shortest libration periods so that saturation begins in a time $P(w) \sim \pi r/|A|w$. Saturation is avoided only if the perturber can drift across the horseshoe zone in a time short compared to $P(w)$, i.e., $w \ll \dot{r}P(w)$.

REFERENCES

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