SHEPHERDING IN A WARM DISK OF LOW OPTICAL DEPTH: A PERTURBATION ANALYSIS: Suguru Araki and William R. Ward, Department of Physics, Hofstra University, Hempstead, NY 11550; Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109; San Juan Capistrano Research Institute, San Juan Capistrano, CA 92675.

Summary. We argue that in a cold planetesimal disk in the protoplanetary nebula the gap formation at the end of incipient runaway growth may not be realized. In a particulate disk of low optical depth such as the protoplanetary disk, disk-planetesimal interactions entail not only angular momentum exchange but also disk heating since the Jacobi constant is conserved between successive encounters. We have performed a perturbation analysis and confirmed our previous numerical analysis that as the disk heats up, torque strength is sustained even though guiding centers of planetesimals are pushed away since they lose angular momentum. However, torque can be maintained only up to a certain limit and further heating will begin to weaken torque. Thus, if the embryo is also subject to nebular tidal torque, it can shift toward the disk edge, leading to continuous accretion of runaway embryos.

Motivation. In the limit of incipient runaway growth planetesimals in the protoplanetary nebula deplete out to the critical accretion range \( d_e = 2\sqrt{3}a(\mu/3)^{1/3} \) on both sides of the orbital radius of a growing protoplanet. This planetary embryo is subject to two kinds of disk tidal torques, one due to the gaseous component and another due to the particulate component. The disk tidal torque from the nebula \( T_d \) tends to decay the orbit of the embryo, where \( T_d = -C_d(\sigma_2 a^2)(a\Omega)^2(a/H)^3 \) and \( C_d \approx O(10^{-1}) \) [1]. The shepherding torque from the planetesimal disk \( T_g \) tries to confine the embryo to a gap where \( T_g = \pm C_d(\sigma_2 a^2)(a\Omega)^2(a/d)^3 \) [1] is the torque due to the particle disk radially separated by \( d \) from the embryo (+ for inner disk, − for outer disk). A slightly simpler situation which involves the essence of the above issue can be considered by focusing on an embryo orbiting outside a disk. In this case the stand-off distance from the disk edge \( d_e \) defined by \( T_g + T_d = 0 \) reads \( d_e = (\sigma_e C_d/\sigma_d C_g)O(10^{-1})H \) which is independent of the embryo mass since both torques have the same mass dependence. When can the shepherding torque keep the embryo at a distance greater than its critical accretion range \( d_e > d_e \)? In other words, find the condition that the embryo behaves as a “shepherd” for the inner particle disk. This requires that the particle disk torque coefficient be larger than a certain value: \( C_d > 8\sqrt{3}C_d(\sigma_e/\sigma_d)(a/H)^{1/3} \). In the solar nebula at \( a \sim 5 \) AU (Jovian zone), \( H \sim 0.35 \) AU, \( \sigma_e/\sigma_d \sim 10^{-2} \), which leads to \( C_d > O(M_p/M_\oplus) \). However, it has been shown that \( C_d \sim 0.83 \) for \( e = 0 \) particle disk [2]. Also, the runaway growth limit is known to produce an embryo mass of \( M_p \sim 2M_\oplus \) for a minimum-mass nebula [3]. These results suggest that the embryo may behave as a “predator” rather than a shepherd as the embryo’s stand-off distance is shorter than its critical accretion range \( d_e < d_e \), so that it will continue to strip off particles from the disk. However, the embryo also heats the disk. The above conclusion is valid only if \( C_d < O(M_p/M_\oplus) \) as the disk heats. What happens to \( C_d \) if disk is heated? We study this issue by perturbation analysis and compare it with our own numerical results [4].

Calculations. It is well known [2] that when a particle on a circular orbit passes by a perturber, epicyclic motion is generated with eccentricity

\[
\epsilon_{GT} = \frac{1}{3} \gamma \mu \left( \frac{a}{b_0} \right)^2
\]

where \( \gamma = (8/3)[K_1(2/3) + 2K_0(2/3)] \sim 6.719 \) and \( b_0 \) is the impact parameter of the incident particle in a cold disk \( (e = 0) \). By virtue of the Jacobi constant, the impact parameter is also increased and there is an exchange of angular momentum. If the particle is part of a ring of high optical depth \( \tau \), the high collision frequency \( \sim \tau \Omega \) will damp eccentricities. At the next encounter the particle will be farther away and suffer a smaller perturbation. The torque decays as the particle separation increases. If the optical depth is low, the eccentricity will not decay between encounters. We want to know how a finite \( e \) affects the angular momentum exchange. We have calculated the perturbation solution of Hill’s equations to first order in the perturber’s mass \( M_p \) and the initial eccentricity \( e \). The post-encounter eccentricity of a warm inner disk particle is given by

\[
\epsilon_{AW}^2 = \left[ \epsilon \cos \phi - \frac{1}{3} \alpha \mu \epsilon \left( \frac{a}{b} \right)^3 \sin \phi \right]^2 + \left[ \epsilon \sin \phi - \frac{1}{3} \beta \mu \epsilon \left( \frac{a}{b} \right)^3 \cos \phi - \frac{1}{3} \gamma \mu \left( \frac{a}{b} \right)^2 \right]^2
\]
where $\alpha = \beta - 4 = (8/9)[19K_1(4/3) + 20K_0(4/3)] - 2 \sim 8.715$ and $\phi$ is the epicyclic phase of the approaching particle. If we assume that the ring is composed of many particles with randomly distributed phase, we can obtain the average $<\Delta(e^2)> = <\frac{1}{3}w_0 - e^2> = \frac{1}{3}\mu(a/b)^2[1 + \chi(ea/b)^2]$ where $\chi = (\alpha^2 + \beta^2)/(2\gamma^2) \sim 2.632$. We want to compare this result with particles of the same Jacobi constant initially in circular orbits. If we assume that there is no dissipation between successive encounters in our tenuous disk, conservation of the Jacobi constant leads to $b^2 = b_0^2 + (4/3)(ea)^2$, where the incident particle would have the impact parameter $b_0$ if the disk were cold: $e = 0$. Substituting this into the above expression and expanding in $e$ gives $<\Delta(e^2)> = \frac{1}{3}w_0 + 8\mu(ea/b_0)^2 + O[(ea/b_0)^4]$. Evaluating the coefficient of the $e^2$ term gives a very small number so that this term nearly vanishes: $\delta = \chi - (8/3) = -0.0348$. Therefore, the change in $e^2$ is almost the same as for the $e = 0$ case even though the particles are farther away. On the other hand, the torque does not increase either so that our requirement for the predator behavior $C_d < O(M_p/M_e)$ seems secure also in a warm disk. However, our perturbation procedure is only valid for $ea \ll b$. To explore the behavior when these quantities are comparable, we have resorted to numerical integration of Hill’s equations [4]. Torque is initially nearly constant as $e$ increases in agreement with perturbation solution. However, eventually torque begins to decrease as shepherding action continues to displace the guiding centers of the particles’ orbits away from the perturber (Figure 1).

Conclusions. Although we have not actually calculated the time variation of the disk torque, our analysis reveals important aspects of its behavior which can be applied to our embryo confinement problem. If the embryo occupies a gap in an optically thin disk such as the planetesimal disk, shepherding action will still take place, but is accompanied by a heating of the disk. This is required since the absence of dissipation in such a disk renders the Jacobi integral constant. The increased eccentricities change the nature of the angular momentum exchange. As a result, the torque strength is sustained even though the guiding centers of the inner disk particles are shepherded away by the loss of angular momentum (outer disk particles gain angular momentum). However, torque can be maintained only up to $e \sim O(b/a)$. Further heating will begin to weaken the torque and confinement strength breaks down. If the embryo is subject to an additional torque due to the nebular component, it can shift closer to the edge of the inner disk (in terms of the Jacobi constant) and begin to accrete particles. This indicates that isolation of runaway embryos may not occur in the Jovian zone for a minimum-mass nebula.


Figure 1:
Total torque exerted on an embryo by a collection of 100 ringlets. Initial eccentricity is same for all ringlets. For $e^*=0$, ringlets spread uniformly over guiding center distances $b^*=[-8.464,-3.464]$. 