ESTIMATING THE AVERAGE DIAMETER OF A POPULATION OF SPHERES FROM OBSERVED DIAMETERS OF RANDOM TWO-DIMENSIONAL SECTIONS.

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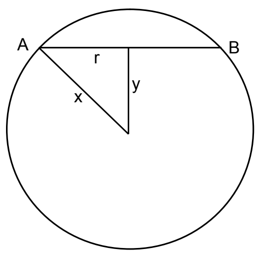
Introduction: Size distributions of chondrules, volcanic fire-fountain or impact glass spherules, or of immiscible globules in silicate melts (e.g., in basaltic mesostasis, agglutinitic glass, impact melt sheets) are imperfectly known because the spherical objects are usually so strongly embedded in the bulk samples that they are nearly impossible to separate. Hence, measurements are confined to two-dimensional sections, e.g. polished thin sections that are commonly examined under reflected light optical or backscattered electron microscopy. Three kinds of approaches exist in the geologic literature for estimating the mean real diameter of a population of 3D spheres from 2D observations: (1) a stereological approach with complicated calculations [1,2]; (2) an empirical approach in which independent 3D size measurements of a population of spheres separated from their parent sample and 'their' 2D cross sectional diameters in thin sections have produced an array of somewhat contested conversion equations [e.g., 3-7]; and (3) measuring pairs of 2D diameters of upper and lower surfaces of cross sections each sphere in thin sections using transmitted light microscopy [8]. We describe an entirely probabilistic approach and propose a simple factor of 4/π (≈ 1.27) to convert the 2D mean size to 3D mean size.



Theoretical Considerations: We assume that the distance 'y' between the center of a given sphere and a random plane (AB in figure above) that intersects it has the uniform distribution on $[-x, x]$ where x is the radius of the sphere. Let r denote the radius of the circle of intersection (fig. above). Then the probability density of r , for a sphere of radius x , is

$$h(r;x) = \frac{r}{x\sqrt{x^2 - r^2}} \quad 0 < r < x. \tag{1}$$

The moments of this distribution are

$$\int_0^x r^k h(r;x)dr = x^k \int_0^x \left(\frac{r}{x}\right)^k \left(\frac{r}{x}\right)\cdot\left(\frac{1}{x}\right)\cdot\frac{1}{\sqrt{1-r^2/x^2}}dr$$

$$= x^k \int_0^1 \frac{u^{k+1}}{\sqrt{1-u^2}}\,du = x^k \int_0^{\pi/2} \sin\theta^{k+1}\,d\theta \qquad \left[u = \frac{r}{x}\right]$$

$$= a_k x^k, say, \quad a_k := \int_0^{\pi/2} \sin^{k+1}\theta \; d\theta \quad (k \ge 1). \tag{2}$$

Let $g(x)$ denote the probability density of the radius x of the sphere. Then the unconditional moments of r are

$$m_k = a_k \int_0^\infty x^k g\left(x\right) dx = a_k \mu_k, \tag{3}$$

$$\mu_k = \frac{m_k}{a_k} \qquad (k \ge 1)$$

where, μ_k is the $k-th$ moment of the distribution of x. An approximate solution from this entirely theoretical approach is $a_1 = \frac{\pi}{4} \approx \cdot 8$, i.e., the average of the radii of observed 2D sections of a set of spheres is about 0.8 times the average of the radii of the spheres.

Solution: We have taken the 2D size distribution data of Fe^0 globules in agglutinitic glass of three lunar soils from Apollo 15, 16, and 17 missions [9] to obtain 'independent' numerical solutions. The soils are of different maturity, from different provenance, and geographically far apart from each other. The raw counts were plotted as standard cumulative curves from which numbers of observations in eight size classes were read off with midpoints at 6.25nm, 18.75nm, 375nm, 87.5nm, 187.5nm, 312.5nm, 437.5nm, and 562.5nm for our data analysis. Tests showed that a gamma size distribution model fits the observed distribution better than a log normal size distribution model (using Pearson's chi square statistic). Tests also showed that our initial assumption of y being distributed uniformly between –x and x produced least errors. Thus, we may assume,

$$g(x) = \frac{1}{\Gamma(\beta)\alpha^{\beta}} x^{\beta-1} e^{-x/\alpha}, \; 0 < x < \infty, \tag{4}$$

where $\alpha > 0, \beta > 0,$ are parameters to be estimated. Maximum Likelihood Estimation (MLE) is the optimal method of estimation of parameters if it can be implemented. For the gamma distribution, we need to solve

$$\frac{\partial}{\partial \alpha} \sum_{i=1}^{n} \log f(r_i) = 0, \quad \frac{\partial}{\partial \beta} \sum_{i=1}^{n} \log f(r_i) = 0, \tag{5}$$

where, $f(r)$ is the marginal density of r, *i.e.*,

$$f(r) = \int_r^\infty h(r;x) g(x) dx = \frac{1}{\Gamma(\beta)\alpha^{\beta}} \int_r^\infty \frac{r}{x\sqrt{x^2 - r^2}} e^{-x/\alpha} x^{\beta-1} dx.$$

We were able to obtain a solution by using the Newton-Raphson iteration in the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}^{(k+1)} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^{(k)} - \frac{1}{\left(\frac{\partial^2 L}{\partial \alpha^2}\right)\left(\frac{\partial^2 L}{\partial \beta^2}\right) - \left(\frac{\partial^2 L}{\partial \alpha \partial \beta}\right)^2} \begin{bmatrix} \frac{\partial^2 L}{\partial \beta^2} & -\frac{\partial^2 L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha^2} \end{bmatrix} \begin{pmatrix} \frac{\partial L}{\partial \alpha} \\ \frac{\partial L}{\partial \beta} \end{pmatrix} \Bigg|_{(\alpha,\beta) = \left(\alpha^{(k)}, \beta^{(k)}\right)}$$

We must note that the iteration, using MAPLE®**,** takes considerable computer time and memory.

Results: Mean 2D and estimated 3D mean sizes of gamma size distributions of the Fe^0 globules in these soils are as follows:

Sample	n (globules)	Mean (2D nm)	Mean (3D nm)	Model (3D mean)
15601	1160	101	123	129
61181	2526	74	87	94
75081	681	208	263	264

The correspondence between MLE and theoretically modeled values is remarkable. We are in the process of modeling the size distributions of Fe^0 globules in 9 other lunar soils. We expect similar results and, if so, to propose a factor of $4/\pi$ ($\approx$1.27) to convert 2D mean size of random sections of spheres to obtain their 3D mean size. Interestingly, this factor is very close to 1.25 that Hughes [6] obtained from a geometric analysis of empirical data.

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