SPHERICAL SPLINES: BEYOND SPHERICAL HARMONICS FOR NON-UNIFORM GEOPHYSICAL DATASETS ON THE MOON AND MERCURY. J. Andreas Ritzer[1], Steven A. Hauck, II[1] and Catherine L. Johnson[2,3], [1]Case Western Reserve University, Cleveland, OH 44106-7216 (andreas.ritzer@case.edu), [2]Scripps Institution of Oceanography, University of California San Diego, San Diego, CA. [3]University of British Columbia, Vancouver, BC, Canada (cljohnson@ucsd.edu).

Introduction: Global-scale datasets, preferably with nearly uniform coverage are important elements of many geophysical studies of the planets and their moons. However, uniformly resolved data are not always available, for example, determination of the gravity field of the Earth's moon suffers from this problem because the data depend upon line-of-sight tracking of a spacecraft's radio signal. Due to the synchronous rotation of the Earth's moon, direct determination of the far side gravity field is not possible with a single spacecraft resulting in a dataset with highly spatially variable quality [1]. An analogous situation exists where particular spacecraft orbits (i.e., highly elliptical ones) may also provide non-uniform data coverage or resolution, especially if the data depend upon spacecraft altitude (e.g., gravity, magnetic fields). The necessarily elliptical orbit with periapse limited to the northern hemisphere of Mercury in the future orbital phase of the MESSENGER mission is an example of the latter situation. Higher surface spatial resolution geophysical data will be limited to the northern hemisphere.

Spherical harmonics are commonly used to represent planetary geophysical datasets [e.g. 2-7]. Among the primary advantages of spherical harmonics are their compact and convenient form as well as the availability of powerful spectral analysis techniques.

In contrast to the ideal case, highly non-uniform data distributions reveal disadvantages associated with spherical harmonic representations. Indeed, they best represent data only for an expansion up to a degree and order appropriate for the data that are least resolved. This essentially truncated expansion (relative to the areas with high data resolution) introduces aliasing that results in an inaccurate representation of the data. Further, the relative truncation of the expansion amounts to discarding data from the more resolved regions. Unfortunately, spherical harmonics are relatively inflexible to local variations in data resolution.

The global support basis for spherical harmonics is one source of their disadvantages with highly non-uniform data. However, non-uniformly resolved data may be more accurately represented by a model representation with local support. Local bases, especially those that rely on meshes produced by Delaunay triangulation, are adaptable to large variations in resolution [8]. Notably, interpolation support can be concentrated where resolving power is highest. Furthermore, locally supported basis functions can accommodate non-uniform, incomplete, and regional data distributions [8].

Spherical Basis Splines: Spherical basis splines (B-Splines) have been increasingly used to solve geophysical problems [see 9-12], particularly in seismology. A 2-D model on a sphere with a local cubic B-spline basis is parameterized in terms of geodesic distance on a triangular grid of knot points. The knot points are control points through which the piecewise polynomial (i.e. the spline) must pass. The normalized cubic B-spline functions are centered on N knots points i=1, 2,…, N on the surface of the planet. Each basis function $f_i(\theta,\varphi)$ depends on the distance Δ from the ith knot point with

$$f_i = \begin{Bmatrix} \frac{3}{4}(\Delta/\Delta_i^0)^3 - \frac{3}{2}(\Delta/\Delta_i^0)^2 + 1, & \Delta \leq \Delta_i^0 \\ -\frac{1}{4}(\Delta/\Delta_i^0)^3 + \frac{3}{2}(\Delta/\Delta_i^0)^2 - 3(\Delta/\Delta_i^0) + 2 & \Delta_i^0 \leq \Delta \leq 2\Delta_i^0 \end{Bmatrix}$$

where $2\Delta_i^0$ is the range of support for the ith basis function [13]. The geophysical values on the surface can be represented by the system of equations $c_i(\theta,\varphi) = a_i f_i(\theta,\varphi)$, where c are the data points, a are the model coefficients, and f is the matrix of B-spline functions. This system can be solved directly if knot points are coincident with the data points, i.e. f is square, or with a least squares norm minimization otherwise. Spherical B-splines also have continuous first and second derivatives [9, 13] which are often important for geophysical modeling.

Lunar Data: The Clementine mission provided a global topographic dataset of the Moon. However, direct knowledge of the gravity field data determined by Lunar Prospector and earlier missions from Doppler radio tracking is constrained by line-of-sight considerations. The tidally locked rotation of the Moon prevents direct radio tracking of a single spacecraft on the far side, so the far side gravity field can only be determined through indirect methods [1]. The resolution of the gravity on the nearside is therefore much greater. We use the Goddard Lunar Topography Model 2 (GLTM 2) expanded up to degree and order 72 [5] as the basis for a test dataset that mimics a dichotomy in data resolution between two hemispheres. We construct the test data set from a globally complete set of data so we can control variable resolving power between the two sides.

Preliminary Results: Using the test data, we can compare the relative accuracy of spherical harmonic and B-spline representations of uniformly and non-uniformly distributed data at a range of resolutions.

The RMS misfit between the data and both spherical harmonic and spline representations with for uniformly resolved input data are nearly zero within machine precision, however the convenience and speed of solution for spherical harmonics make it a more likely choice for nearly uniform data.

Our irregular test datasets have a higher data resolution on the nearside of the Moon and a lower resolution data distribution on the far side. Within each hemisphere the data are uniformly distributed.

B-splines with data at knot points. The solution to a B-spline problem with coincident data and knot points is solved exactly as a linear system of equations. Even with a large disparity in resolution, the RMS misfit for the B-spline model is zero, within numerical precision. The solutions for non-uniformly resolved data using spherical harmonics are quite inaccurate for very non-uniform data (Figure 1) as a least-squares solution is necessary and the degree and order to which the model is represented is limited by the lower resolution hemisphere. Thus, for the case where the knot points are placed at each data point, there is a clear advantage to the use of B-splines over spherical harmonics.

B-splines with a spherical tessellation of knots. The number of knot points may be practically limited by computational capability making direct solutions untenable. In this case we use a tessellation of evenly spaced points on a sphere, produced by an optimization method where the sum of $1/r$ between all pairs, r being geodesic distance, is minimized [14].

When data and knot points do not coincide a least-squares solution is implemented. The misfits for non-uniformly resolved data with an evenly spaced distribution with nearly as many knots as data, was very low, yet not quite zero (Figure 1). When a fewer number of knots (appropriate for a lower resolution data set) is used, aliasing is introduced (Figure 1), yet the misfit of the lower resolution tessellation is still less than that of spherical harmonics. This demonstrates that splines on a regular tessellation of knot points can also be more accurate when dealing with non-uniform data distributions.

Knot point distribution. The selection of knot points is the most important factor which determines the accuracy of a B-spline model. Thus, optimal distributions of knot points will produce the most accurate spline representations. An automatic knot point selection algorithm can be used [8] which provides an optimal distribution of knot points fit to local resolution.

Discussion and Future Directions: Spherical splines appear to have a notable advantage in accuracy when dealing with non-uniformly resolved planetary datasets on surfaces and may alleviate some of the disadvantages of spherical harmonic representations. Future work will focus on investigating the suitability of splines for gravity and magnetic modeling, particularly in light of such geophysical datasets' altitude dependence.

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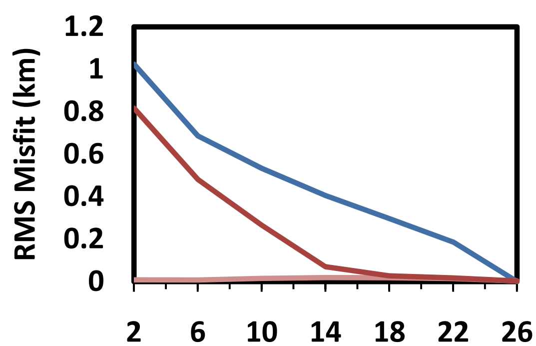

Figure 1: The RMS misfit of both basis models of non-uniformly distributed test data sets, where the degree to which the resolution is appropriate on the farside of the test data is shown on the bottom axis. The nearside resolution was kept constant at a degree of 26. The misfits for all uniform resolution models and direct solutions are nearly zero and are not shown. The top blue line shows the misfit for spherical harmonics. The middle red line is for a low resolution tessellation appropriate for the resolution of the farside. The bottom orange line is for a high resolution tessellation with n-1 knots where n is the number of data points. The maximum relief in our test data was 5.8 km.