Vertical Structure of a Two-Phase Pre-lunar Disk.

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The post giant impact evolution of a two-phase pre-lunar disk is examined, adopting the Thompson & Stevenson argument [1] that the spreading rate of a disk containing of order a lunar mass will be radiation limited, and that the disk’s vertical temperature profile will be regulated by the vapor-melt phase transition.

The equations that govern the structure of a convecting two-phase vapor-liquid disk are [1]: The ideal gas equation,

\[ P = \rho gRT/\mu, \]  

where \( P(z) \) is the pressure, \( T(z) \) is the temperature, \( \rho_g(z) \) is the gas density, \( R = 8.31 \times 10^7 \) ergs/mole-K is the universal gas constant, \( \mu \sim 30 \) g/mole is the effective molecular weight of the vaporized rock and \( z \) is the vertical distance from the mid-plane. The adiabatic equation,

\[ \frac{dP}{dz} = c^2 \frac{d\rho}{dz}, \]  

where \( \rho(z) \) is the total density which is due to both components, i.e.,

\[ 1/\rho = x/\rho_g + (1-x)/\rho, \]  

with \( \rho_l \) being the density of the liquid/solid particles, and \( x, (1-x) \) representing the mass fractions of gas and condensed phases, respectively. The two-phase sound speed [2]

\[ c^2 = \rho_g \rho \left[ (C_gT/l^2)(1-x) + \left( \frac{\mu RT - 2/l + C_sT/l^2}{x} \right)^{-1} \right], \]

where \( C_g \approx (3/2)R/\mu \approx 4.16 \times 10^7 \) ergs/g-K, \( C_s \sim 10^7 \) ergs/g-K is the specific heat of the condensed phase, and \( l = 1.7 \times 10^{11} \) ergs/g is the latent heat of vaporization. The equation of hydrostatic equilibrium,

\[ \rho_1 \frac{dP}{dz} = -z\Omega^2, \]

where \( \Omega \) is the orbital frequency of the disk. The Clausius-Clapeyron equation describing phase equilibrium,

\[ P = P_c \exp(-T_c/T), \]

where \( T_c \equiv \mu l/\sigma R = 6 \times 10^4 \) K and \( P_c = 3 \times 10^{14} \) dynes [3].

Thompson & Stevenson considered a disk with a very small gas mass fraction akin to a "foam", such that \( \rho_g/\rho_c \ll x \ll 1 \) (i.e., with \( x \sim 10^{-3} \) to \( 10^{-2} \)), and assumed that convection would keep such a low-\( x \) disk vertically well mixed. This lowered the sound speed considerably, allowing most of the disk surface density, \( \sigma \), to participate in gravitational instability. However, the authors recognized the potential for grain settling and briefly suggested other possible evolutionary paths, including a settled component that causes the disk in the mid-plane to be more dissipative with a faster spreading time. Such an alternative is pursued here.

Settling of condensates will decrease \( x \) (the vapor mass fraction) in the mid-plane but increase it at altitude. Consequently, we envision a vertically stratified structure, with a mid-plane layer surface density \( \sigma \), composed of high-density magma/solids subject to transient gravitational instabilities, and a thick surrounding atmosphere of vapor in equilibrium with a small amount of grains. For the latter, we seek an alternative vertical disk structure that satisfies the equations when \( x \sim O(1) \). Interestingly, this simplifies the mathematics and allows the system to be approximated analytically. In this case the sound speed increases to \( c = x^{1/2}(\mu RT - 2/l + (3/2)RT/\mu)^{1/2} \sim (xRT/\mu)^{1/2} > c_{crit} \), where \( c_{crit} = \pi G\sigma/\Omega \) is the critical sound speed for gravitational instability, so that the predominately vapor atmosphere is gravitationally stable. Since \( x \) is not small, the first term dominates on the RHS of eqn. (3) and we may approximate \( \rho_g \sim x \rho_c \).

Using this in eqn. (1), \( P \approx \rho x RT/\mu \approx \rho c^2 \), implying the \( P/\rho \approx c^2 \). Substituting this in eqn. (2) gives \( dP/dz = (P/\rho)d\rho/dz \), which we integrate from the mid-plane to deduce \( P/P_c = \rho/\rho_c \) where \( P_c \) and \( \rho_c \) denote central mid-plane values of the pressure and density of the atmosphere at the interface between it and the magma layer. From this we conclude that \( c^2 = P/\rho = P_c/\rho_c = constant \), so that the temperature and gas fraction must vary in such a manner as to keep the sound speed constant throughout the atmosphere, so accordingly,

\[ T/T_c = x_c/x, \]
Making the substitution $\rho = P/c^2$ in the hydrostatic equation (5) and integrating from the mid-plane, $z = 0$, gives

$$P = P_c \exp(-z/H)^2; \rho = \rho_c \exp(-z/H)^2, \quad (8)$$

where the scale height $H \equiv \sqrt{2}c/\Omega$ is introduced. To get the variation of temperature with height, eqn. (6) is employed to replace $P = P_c \exp(-T_o/T_c)$, $P_c = P_o \exp(-T_o/T_c)$ in the first of eqns. (8) to find,

$$T_o/T = T_o/T_c + (z/H)^2. \quad (9)$$

This, together with eqn. (7), also tells us how $x/x_c$ changes. Finally, eqn. (6) is used to set

$$c^2 = P_c/\rho_c = (P_o/\rho_o) \exp(-T_o/T_c). \quad (10)$$

Two boundary conditions are needed to complete the solution. The first of these is supplied by the surface density, $\sigma$, which we assume is of order $\sim M_{moon}/\pi r_K^2$, where $r_K$ is the Roche distance from the Earth. Hence

$$\sigma = \int_{-\infty}^{\infty} \rho dz = \sqrt{\pi} \rho_H = \sqrt{2\pi} \rho_c (c/\Omega). \quad (11)$$

Using this to eliminate $\rho_c$ in eqn. (10) results in

$$c = \sqrt{2\pi} (P_o/\sigma \Omega) \exp(-T_o/T_c). \quad (12)$$

The second comes from setting the temperature equal to the condensation temperature, $T = T_{ph} = 2000K$ at the height of the photosphere $z = z_{ph}$.

Above the photosphere the disk is optically thin and non-convecting. This allows grains to settle and we expect the gas mass fraction $x$ to approach unity at $z_{ph}$. This allows one to immediately evaluate $c = (RT/\mu)^{1/2} \sim 7.44 \times 10^4 \text{ cm/s}$. At the Roche distance, $\Omega \sim 2.4 \times 10^{-5} \text{ s}^{-1}$ and the scale height is $H \sim 4.4 \times 10^8 \text{ cm}$. For a nominal initial disk containing a lunar mass within the Roche limit, the average surface density (condensates + vapor) is $\sigma \sim 6 \times 10^6 \text{ g/cm}^2$. Using this in eqn. (11), yields a mid-plane density $\rho_c \sim 7.7 \times 10^3 \text{ g/cm}^3$ and from eqn. (12) a central temperature of $T_c \sim 3807K$. The central pressure is $P_c = \rho_c c^2 \sim 4.3 \times 10^7 \text{ dynes}$, while the gas mass fraction in the mid-plane decreases to $x_c = T_{ph}/T_c \sim 0.53$. Finally, eqn. (9) tells us the height of the photosphere, $z_{ph} \sim 3.78H$. Note that the critical sound speed for gravitational instability is $c_{s,crit} \sim 5 \times 10^7 \text{ cm/s}$ so that the atmosphere is stable.

In order to maintain the quasi steady-state of the disk, the energy radiated from its photosphere must be replaced. We envision a mid-plane melt layer of surface density $\sigma_s$, where gravitational instabilities generate an effective viscosity

$$\nu_{inst} \sim \zeta \pi^2 G^2 \sigma_s^2/\Omega^3 \quad (13)$$

causing the layer to spread [4,5]. The viscous energy dissipation rate per unit area, $\sim (9/4) \sigma_s v \Omega^2$, is then made comparable to the radiative cooling rate, $\sim 2\sigma_{SB} T_{ph}^4$, by adjusting the surface density of the condensates to a critical value,

$$\sigma_{s,crit} \sim (\Omega \sigma_{SB} T_{ph}^4/\zeta \pi^2 G^2)^{1/3}, \quad (14)$$

where $\sigma_{SB} = 5.67 \times 10^{5} \text{ ergs/cm}^2 \text{s.K}^4$ is the Stefan-Boltzman constant. For $T_{ph} \sim 2000^0K$, $\zeta \sim 1$, $\sigma_{s,crit} \sim 1.7 \times 10^8 \text{ g/cm}^2$; if $\sigma_s < \sigma_{s,crit}$, additional material condenses from the vapor phase, and if $\sigma_s > \sigma_{s,crit}$, material evaporates. It is likely that for a disk massive enough to produce the Moon, most of the pre-lunar material is stored in the vapor reservoir at the beginning of the evolution.

The magma layer spreads on a time scale

$$\tau_s \sim r_K^2/\nu \sim r_K^2 \Omega^3/\zeta \pi^2 G^2 \sigma_{s,crit}^2 \sim 12.6 \text{ yrs} \quad (15)$$

This is the time scale over which material initially spreads beyond the Roche distance and lunar accretion begins. However, as material leaves the Roche interior magma layer, more condensate from the vapor atmosphere replenishes it. Consequently, the overall process of depleting the original magma plus vapor disk takes of order

$$\tau_{red} \sim \sigma(r^2 \Omega^2 + I)/2\sigma_{SB} T_{ph}^4 \sim 40 \text{ yrs}. \quad (16)$$

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References.