GRAVITATIONAL STRESSES IN HYPERION

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Introduction

Hyperion is the largest irregularly shaped satellite in the Solar System, with approximate tri-axial dimensions of 205×130×110 km in radius. The Hyperion craters are particularly deep and provides a curiously punched-in look, somewhat like the surface of a wasp nest (see Image 1). Planetary geologists have theorized that Hyperion's high-porosity and low density would crater more by compression than excavation. Many of the crater walls on Hyperion are bright, which suggests an abundance of water ice with a bulk density $\rho_0 = 0.544$ g/cm^3 [MH].

Analytical procedure

Unfortunately, none of the elegant spherical theory [HJ] can be applied to Hyperion because of its huge eccentricity ε = 0.812. On the other hand, Hyperion is tolerably well approximated by a two-axial prolate ellipsoid of principal semiaxes a = 205 km and b = 119,6 rm. So, the Saturnian satellite is modeled as homogeneous, elastic two-axial ellipsoid subject to self-gravitational stress. An exact analytical treatment then gives the stress and strain fields throughout its interior. Applications of the new formulation to other nonspherical bodies in the solar system are also discussed.

The gravitational potential V of a two-axial ellipsoid is rather simple in the interior [DN] and, if the density ρ_0 is assumed constant, can be written as

$V / D_0 = a^2 A(\varepsilon) - z^2 B(\varepsilon) - (x^2 + y^2)\, C(\varepsilon)$,

where (x,y,z) are Cartesian coordinates directed along the (b,b,a) axes respectively; A,B,C are functions of ε have been found analytically

$A(\varepsilon) = \ln((1+\varepsilon)/(1-\varepsilon))/(2\varepsilon)$,

$B(\varepsilon) = \ln((1+\varepsilon)/(1-\varepsilon))/(2\varepsilon^3) - 1/\varepsilon^2$,

$C(\varepsilon) = 1/(2\varepsilon^2 (1-\varepsilon^2)) - \ln((1+\varepsilon)/(1-\varepsilon))/(4\varepsilon^3)$,

$D_0 = 2\pi\,(1-\varepsilon^2)\,\rho_0\, G$, $\varepsilon^2 = 1 - b^2/a^2$

and *G* is the Newtonian constant of gravitation.

Describe the deformation by a vector field $u_i(x_i)$, where x_1 = x, x_2 = y, x_3 = z, and u_i is the displacement in the x_i direction. If the distortions are presumed small, Hooke's low (connection between stress and strain tensors) yields

$(\lambda + \mu)\, \partial^2 u_j / \partial x_i \partial x_j + \mu \partial^2 u_i / \partial x_j^2 = -\rho_0 \partial V/\partial x_i$,

where λ and μ are the two Lame parameters. μ is equal to the shear modulus, or rigidity, while λ is a measure of incompressibility related to Poisson's ratio v by

$\lambda = \mu(2v) / (1-2v)$

In general this equation is difficult to integrate for u_i and numerical calculations for specific planetary bodies were used [DB]. But, because of the simple structure of the above gravitational potential and the following additional conditions:

1. surface of the body is free. It means that the stress normal and tangential to the surface must vanish;
2. the body remains continuous without gaps and cracks

analytical solution for u_i by means of fourth degree polynomials of x, y, z is possible.

Results

The most simple forms have two combination of u_i which posses a clear physical meaning

$P = -1/3\, \sigma_{kk} = P_0\, ((1+v)/3(1-2v))\, [A_0(\varepsilon) + B_0(\varepsilon)x^2 + C_0(\varepsilon)y^2 + E_0(\varepsilon)z^2]$ - hydrostatic pressure;

$\omega^2 = z^2(x^2+y^2)K_0(\varepsilon)$ - ω is rotation angle of any small volume near point (x,y,z) in the plane containing Z-axis;

where A_0, B_0, C_0, E_0, K_0 are combinations of A(ε),B(ε),C(ε), $P_0 = 2\pi\, \rho_0^2 b^2\, G$.

Analytical solution illustrates an important point: the stress tensor is independent of the rigidity of the material and depends Poisson's ratio and eccentricity, only.

Thereafter only an equatorial cross section (x,y,0) from the general solution is analyzed as this seems to be the most interesting slice. The xz and yz sections contain no great surprises. In this slice z = 0 and polar radius r as $r^2 = x^2 + y^2$ can be defined. Then, radial strain tensor is

$D_{rr} = \partial u_r / \partial r$

and dimensionless expression depending r/b is shown on Fig.1, where v = 0.31 as for water ice. The line of strokes corresponds ε = 0.1 (almost sphere), the dotted line - ε = 0.4 and heavy line - ε = 0.8 (ellipsoid similar Hyperion). Note, that inside r/b ≈ 0.7 ice is compressed - $D_{rr} < 0$ and outside is stretched - $D_{rr} > 0$. For sphere ε = 0 and the above threshold $(r/b)^2 = (3 - v)/(3(1 + v))$ for water ice is about r/b ≈ 0.827

Hydrostatic pressure attains a peak value P = 0.348 P_0 ≈ 0.621 MPa or 6.21 bar at the center of Hyperion, falling off to P = 0.088 P_0 ≈ 0.157 MPa or 1.57 bar at the surface (see Fig.2). Materials are known to fail under pressure if realistic criterion named for Tresca is met [HJ]. It states that solids fracture along surfaces halfway between the axes of greatest stress σ_1 and least stress σ_3 whenever the maximum shear stress τ_{max} exceeds some constant S_0 characteristic of the material known as its *shear strength:*

$\tau_{max} = (\sigma_1 - \sigma_3)/2 > S_0$

Dimensionless expression of τ_{max} depending r/b is shown on Fig.3. Maximum shear stress attains a peak value P = 0.088 P_0 ≈ 0.157 MPa or 1.57 bar at the surface of Hyperion, falling off to P = 0.045 P_0 ≈ 0.08 MPa or 0.8 bar at the center.

Following more detailed analytical analysis [VS] will show that the ratio of maximum share stress (for prolongate water ice ellipsoid like Hyperion) on the pole – z = a, x,y = 0 and on the equator – z = 0, x,y = b is

$\tau_{max}(\text{pole}) / \tau_{max}(\text{equator}) \approx 0.36$

Discussion

Experimental data for pure ice define an upper limit of shear strength S_0 at 203 K as

$S_0 < 2$ MPa,

Because of the spongy Hyperion, we can set lower limit of S_0 for pure ice as

$S_0 > 0.157$ MPa

with the certainty now. Also, the above consideration specifies results of the semi analytical approach [SV] and provides more confident data for ice planetary bodies.

So, our analytical study had shown that for the small planetary bodies with huge eccentricity, spherical theory is unusable base. There may be a discrepancy calculated values of stress tensors in 2-3 times.

References

[MH] Hicks, M. et.al. (2008) *Icarus*, 193, 352-358 [HJ] Jeffreys, H. (1959), ***The Earth. Its origin, history and physical constitution***. Cabridge, University Press [DN] Danby, J. M. A. (1962). ***Fundamentals of Celestial Mechanics.*** Macmillan Co., New York [DB] Dobrovolskis, A. (1982) *Icarus*, 52, 136-148 [VS] Voropaev, S., Slyuta, E. Under publication [SV] Slyuta, E., Voropaev, S. (1997) *Icarus*, 129, 401-414

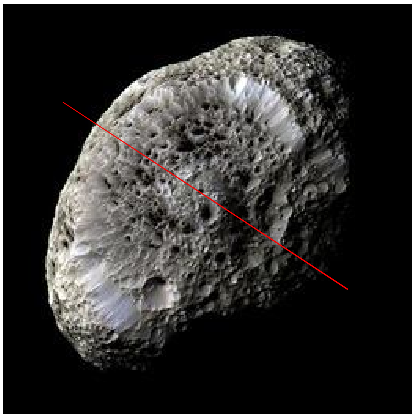

Image1. Spongy Hyperion
NASA's Cassini spacecraft obtained this unprocessed image of Saturn's moon Hyperion on Sep. 26, 2005.

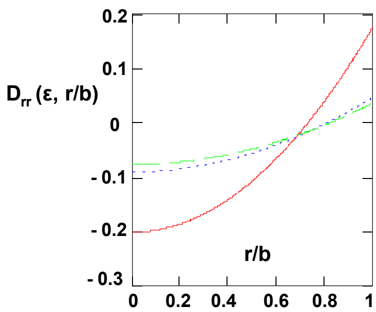

Fig.1 Radial strain tensor in equatorial slice

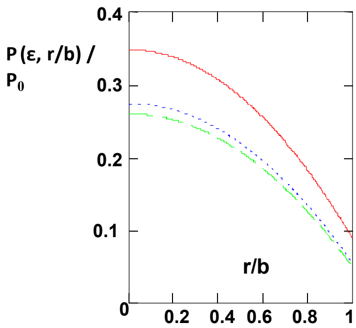

Fig.2 Hydrostatic pressure in equatorial slice

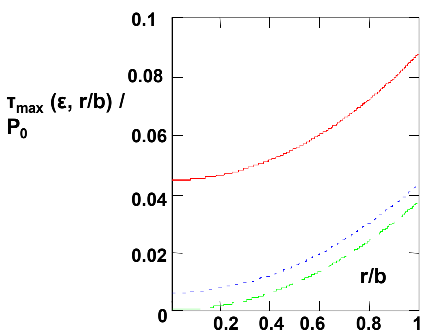

Fig.3 Max. shear stress in equatorial slice