GRAVITATIONAL STRESSES IN HYPERION

Sergey A. Voropaev
Vernadsky Institute of Geochemistry and Analytical Chemistry RAS

Introduction

Hyperion is the largest irregularly shaped satellite in the Solar System, with approximate dimensions of 205x130x110 km in radius. The Hyperion craters are particularly deep and provides a curiously punched-in look, somewhat like the surface of a wasp nest (see Image 1). Planetary geologists have theorized that Hyperion's high porosity and low density would crater more by compression than excavation. Many of the crater walls on Hyperion are bright, which suggests an abundance of water ice with a bulk density \( \rho = 0.544 \text{ g/cm}^3 \) [MH].

Analytical procedure

Unfortunately, none of the elegant spherical theory [HJ] can be applied to Hyperion because of its huge eccentricity \( e \approx 0.812 \). On the other hand, Hyperion is tolerably well approximated by a two-axial prolate ellipsoid of principal semiaxes \( a = 205 \text{ km} \) and \( b = 119.6 \text{ km} \). So, the Saturnian satellite is modeled as homogeneous, elastic two-axial ellipsoid subject to self-gravitational stress. An exact analytical treatment then gives the stress and strain fields throughout its interior. Applications of the new formulation to other nonspherical bodies in the solar system are also discussed.

The gravitational potential \( V \) of a two-ellipsoid is rather simple in the interior [DN] and, if the density \( \rho_0 \) is assumed constant, can be written as:

\[
V = \frac{G M}{r} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \frac{1}{2} \left( 1 - \frac{r}{a} \right)^2 \left( 1 - \frac{r}{b} \right)^2 \left( 1 - \frac{r}{c} \right)^2
\]

where \( M = \rho_0 V \) is mass, \( G = 6.67 \times 10^{-11} \text{ N m}^2 / \text{kg}^2 \) is gravitational constant, \( c = c_1 = c_2 = \) constant, where \( c_1 \) and \( c_2 \) are Cartesian coordinates directed along the \((b,b,a)\) axes respectively, \( A,B,C \) are functions of \( \epsilon \) that have been calculated analytically [MH].

The gravitational stress is:

\[
\sigma_{ij} = \frac{\partial V}{\partial x_i} \delta_{ij} - \frac{1}{2} \frac{\partial^2 V}{\partial x_i \partial x_j}
\]

This equation is difficult to integrate for \( \epsilon \), but if \( \epsilon \) is small, Hooke’s low (connection between stress and strain tensors) yields

\[
\lambda = \mu \frac{2 \nu}{(1 - 2 \nu)} \frac{\partial^2 V}{\partial x_i \partial x_j}
\]

where \( \lambda \) and \( \mu \) are the two Lame parameters. \( \mu \) is equal to the shear modulus, \( \lambda \) to the bulk modulus.

Results

The most simple forms have two combination of \( \psi \) which possess a clear physical meaning:

\[
P = \frac{1}{3} \epsilon_0 = \frac{P_0}{2} (1 + \psi)^3 (1 - 2 \psi) \left[ A_1 (\epsilon) + B_1 (\epsilon) + C_1 (\epsilon) \right] - \frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2 V}{\partial x_i^2}
\]

where \( A_1, B_1, C_1, K_1 \) are combinations of \( A(\epsilon), B(\epsilon), C(\epsilon) \). Analytical solution is an important point: the stress tensor is independent of the rigidity of the material and depends Poisson's ratio \( v \).

Thereafter only an equatorial cross section (x,y,0) from the general solution is analyzed as this seems to be the most interesting slice. The xz and yz sections contain no great surprises. In this slice \( z = 0 \) and on the equator \( x = 0, y = b \) is

\[
\sigma_{xx} = \sigma_{yy} = \frac{1}{2} \frac{\partial^2 V}{\partial x_i^2} \left( 1 + \frac{1}{2} \frac{\partial^2 V}{\partial x_i^2} \right)
\]

Dimensionless expression for \( \sigma_{xx} \) depends on the radius \( r \) of Hyperion and

\[
0.2 \leq \frac{r}{b} \leq 1
\]

Following more detailed analytical analysis [VS] will show that the ratio of maximum shear stress (for prolongate water ice ellipsoid like Hyperion) on the pole – \( z = a, x = 0 \) and on the equator – \( z = 0, x = b = \) real(pole) / real(equator) = 0.36

Discussion

Experimental data for pure ice define an upper limit of shear strength \( S_y \) at 203 K as \( S_y < 2 \text{ MPa} \). Because of the spongy Hyperion, we can set lower limit of \( S_y \) for pure ice as \( S_y > 0.157 \text{ MPa} \). With the certainty now. Also, the above consideration specifies results of the semi analytical approach [SV] and provides more confident data for ice planetary bodies.

So, our analytical study had shown that for the small planetary bodies with huge eccentricity, spherical theory is unusable base. There may be a discrepancy calculated values of stress tensors in 2-3 times.

References


Image 1. Spongy Hyperion
NASA’s Cassini spacecraft obtained this unprocessed image of Saturn’s moon Hyperion on Sep. 25, 2005.