Introduction: Hyperion is the largest irregularly shaped satellite in the Solar System, with approximate tri-axial dimensions of $205 \times 130 \times 110$ km in radius. The Hyperion craters are particularly deep and provides a curiously punched-in look, somewhat like the surface of a wasp nest (see Fig. 1). Planetary geologists have theorized that Hyperion’s high-porosity and low density would crater more by compression than excavation. Many of the crater walls on Hyperion are bright, which suggests an abundance of water ice with a bulk density $\rho_0 = 0.544$ g/cm$^3$ [MH].

Analytical procedure: Unfortunately, none of the elegant spherical theory [HJ] can be applied to Hyperion because of its huge eccentricity $\varepsilon = 0.812$. On the other hand, Hyperion is tolerably well approximated by a two-axial prolate ellipsoid of principal semi axes $a = 205$ km and $b = 119.6$ rm. So, the Saturnian satellite is modeled as homogeneous, elastic two-axial ellipsoid subject to self-gravitational stress. An exact analytical treatment then gives the stress and strain fields throughout its interior. Applications of the new formulation to other nonspherical bodies in the solar system are also discussed.

The gravitational potential $V$ of a two-axial ellipsoid is rather simple in the interior [DN] and, if the density $\rho_0$ is assumed constant, can be written as

$$ V / D_0 = a^2 A(\varepsilon) - z^2 B(\varepsilon) - (x^2 + y^2) C(\varepsilon), $$

where $(x, y, z)$ are Cartesian coordinates directed along the $(b, b, a)$ axes respectively; $A, B, C$ are functions of $\varepsilon$ have been found analytically

$$ A(\varepsilon) = \ln((1 + \varepsilon)/(1 - \varepsilon))/(2\varepsilon), $$

$$ B(\varepsilon) = \ln((1 + \varepsilon)/(1 - \varepsilon))/\varepsilon^2 - 1/\varepsilon^2, $$

$$ C(\varepsilon) = 1/(2\varepsilon^2 (1 - \varepsilon^2)) - \ln((1 + \varepsilon)/(1 - \varepsilon))/(4\varepsilon^3), $$

$$ D_0 = 2\pi (1 - \varepsilon) \rho_0 G. \varepsilon^2 = 1 - b^2/a^2 $$

and $G$ is the Newtonian constant of gravitation.

Describe the deformation by a vector field $u_i(x_i)$, where $x_i = x, y, z$, and $u_i$ is the displacement in the $x_i$ direction. If the distortions are presumed small, Hooke’s low (connection between stress and strain tensors) yields

$$ (\lambda + \mu) \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_j} = - \rho_0 \frac{\partial V}{\partial x_i}, $$

where $\lambda$ and $\mu$ are the two Lame parameters. $\mu$ is equal to the shear modulus, or rigidity, while $\lambda$ is a measure of incompressibility related to Poisson’s ratio $\nu$ by

$$ \lambda = (\mu(2\nu)/(1 - 2\nu). $$

In general this equation is difficult to integrate for $u_i$ and numerical calculations for specific planetary bodies were used [DB]. But, because of the simple structure of the above gravitational potential and the following additional conditions

1. the surface of the body is free. It means that the stress normal and tangential to the surface need vanish,
2. the body remains continuous without gaps or cracks.

analytical solution for $u_i$ by means of fourth degree polynomials of $x, y, z$ is possible.

Results: The most simple forms have two combination of $u_i$ which poses a clear physical meaning $P = \sigma_{22} = P_0 ((1 + \nu)/(3(1 - 2\nu)) [A_0(\varepsilon) + B_0(\varepsilon)x^2 + C_0(\varepsilon)y^2 + E_0(\varepsilon)z^2] -$ hydrostatic pressure

$$ \omega = (x^2 + y^2)K_0(\varepsilon) - \omega = \text{rotation angle of any small volume near point } (x, y, z) \text{ in the plane containing } Z \text{-axis}; $$

where $A_0, B_0, C_0, E_0, K_0$ are combinations of $A(\varepsilon), B(\varepsilon), C(\varepsilon), P_0 = 2\pi \rho_0 b^2 G$. Analytical solution illustrates an important point: the stress tensor is independent of the rigidity of the material and depends Poisson’s ratio and eccentricity, only.

Thereafter only an equatorial cross section $(x, y, 0)$ from the general solution is analysed as this seems to be the most interesting slice. The $xz$ and $yz$ sections contain no great surprises. In this slice $z = 0$ and polar radius $r$ as $r^2 = x^2 + y^2$ can be defined. Then, radial strain tensor is

$$ D_r = \frac{\partial u_r}{\partial r}$$

dimensionless expression depending $r/b$ is shown on Fig.1, where $v = 0.31$ as for water ice. The line of strokes corresponds $\varepsilon = 0.1$ (almost sphere), the dotted line - $\varepsilon = 0.4$ and heavy line - $\varepsilon = 0.8$ (ellipsoid similar Hyperion). Note, that inside $r/b = 0.7$ ice is compressed - $D_r < 0$ and outside is stretched - $D_r > 0$. For sphere $\varepsilon = 0$ and the above threshold $\varepsilon^2 = 3 - v/(4(1 + v))$ for water ice is about $r/b \approx 0.827$

Hydrostatic pressure attains a peak value $P = 0.348$ $P_0 \approx 0.621$ MPa or 6.21 bar at the center of Hyperion, falling off to $P = 0.088$ $P_0 = 0.157$ MPa or 1.57 bar at the surface (see Fig.2). Materials are known to fail under pressure if realistic criterion named for Tresca is met [HJ]. It states that solids fracture along surfaces halfway between the axes of greatest stress $\sigma_1$ and least stress $\sigma_3$ whenever the maximum shear stress $\tau_{\text{max}}$ exceeds some constant $S_0$ characteristic of the material known as its shear strength:

$$ \tau_{\text{max}} = (\sigma_1 - \sigma_3)/2 > S_0 $$

Dimensionless expression of $\tau_{\text{max}}$ depending $r/b$ is shown on Fig.3. Maximum shear stress attains a peak
value $P = 0.088 P_0 \approx 0.157 \text{ MPa}$ or 1.57 bar at the surface of Hyperion, falling off to $P = 0.045 P_0 \approx 0.08 \text{ MPa}$ or 0.8 bar at the center. Following more detailed analytical analysis [VS] will show that the ratio of maximum share stress (for prolongate water ice ellipsoid like Hyperion) on the pole $-z = a, x, y = 0$ and on the equator $-z = 0, x, y = b$ is

$$\frac{\tau_{\text{max}}(\text{pole})}{\tau_{\text{max}}(\text{equator})} \approx 0.36$$

**Discussion:** Experimental data for pure ice define an upper limit of shear strength $S_0$ at 203 K as

$S_0 < 2 \text{ MPa}$

Because of the spongy Hyperion, we can set lower limit of $S_0$ for pure ice as

$S_0 > 0.157 \text{ MPa}$

with the certainty now. Also, the above consideration specifies results of the semianalytical approach [SV] and provides more confident data for ice planetary bodies.

**Conclusions:** Our analytical study had shown that for the small planetary bodies with huge eccentricity, spherical theory is unusable base. There may be a discrepancy calculated values of stress tensors in 2-3 times.


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**Fig. 1. Spongy Hyperion**

NASA’s Cassini spacecraft obtained this unprocessed image of Saturn’s moon Hyperion on Sep. 26, 2005.